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A THEORETICAL ANALYSIS OF ELASTIC VIBRATIONS OF FIXED-ENDED AND HINGED HELICOPTER BLADES IN HOVERING AND VERTICAL FLIGHT

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SUMMARY

A theoretical analysis is given of the frequency and damping characteristics of the free modes of vibrations of balanced fixed-ended and hinged elastic helicopter rotor blades in hovering and vertical flight. Torsional vibrations, bending vibrations in flapping and in lagging, and coupling between the flapping and lagging motions are considered. Explicit methods and formulas for the calculation of natural frequencies and logarithmic decrements of the principal modes are developed, from which general conclusions are rigorously drawn. Flutter of helicopter blades, which may occur when the blades are unbalanced, is briefly considered on the basis of quasi-stationary flow, and simple criterions are derived for the stability of the coupled torsional and flapping vibrations in such cases.

INTRODUCTION

The purpose of this investigation is to present under one cover, in a brief and simple fashion, a comprehensive analysis of the frequency and damping characteristics of elastic helicopter rotor blades in hovering and vertical flight when they perform small vibrations about a state of static equilibrium. Effects which appear not to have been thoroughly investigated heretofore are considered in detail here. These include: Consideration of the effect of boundary conditions on the centrifugal contributions to the natural vibration frequencies of rotating blades in bending; the effect of free torsional vibrations on the flapping and lagging vibrations of mass-balanced blades; the effect of Coriolis, centrifugal, and aerodynamic coupling between the flapping and lagging vibrations of a helicopter blade; and a comparison between aerodynamic and internal damping in the principal modes of rotating beams in bending and in torsion.

The analysis and general formulas have been developed for blades having any taper and cross-sectional distribution, with the restriction that the shear center and the center of gravity of a cross section coincide (as is usually the case, for example, for a single-tubular-spar section). Moreover, the case of mass-balanced blades, that is, blades whose cross-sectional centers of gravity coincide with the aerodynamic centers, has been emphasized, since attempts are usually made in practice to achieve such a condition. (See reference 1.) Nevertheless, the vibrations of unbalanced blades, characterized by coupling between flapping and torsion with the resulting possibility of flutter, have been briefly analyzed.

In the case of hinged blades, it has been assumed that the flapping (horizontal) and lagging (vertical) hinge axes are intersecting and perpendicular to the blade axis, so that no change in pitch angle of a blade is caused by either flapping or lagging.

The general procedure in this analysis consists first in setting up the equations of motion of a rotating helicopter rotor blade in hovering and vertical flight in torsion and in bending in two mutually perpendicular directions (flapping and lagging) and then in solving these equations either exactly or else approximately by the Rayleigh-Ritz method.

Unless otherwise stated, the details of the mathematical derivations have been almost entirely omitted here for the sake of brevity; however, these details have been worked out by the present author in an unpublished paper entitled "A Theoretical Analysis of the Elastic Vibrations of Fixed-Ended Helicopter Blades in Flight."

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SYMBOLS

A	cross-sectional area of a blade
A_0	value of A at root of a blade
B	downwash factor, equation (2b)

C	dimensionless parameter $\left(\pi \frac{\rho}{\sigma} \frac{lc_0}{A_0}\right)$
c	length of chord of a blade section
c_0	value of c at root of blade
$\left(c_{d_0}\right)_1$	value of profile-drag coefficient of blade section when absolute angle of attack is equal to $\theta_0 + \theta_1$
$\left(c_{d_0}\right)_0$	value of profile-drag coefficient of blade section at zero absolute angle of attack
d	dimensionless distance between midpoint and shear center (here, also center of gravity) of a blade section (fig. 1)
E	modulus of structural blade material
e	distance between point of attachment of blades and rotor axis of rotation; eccentricity
G	shear modulus of structural blade material
g	acceleration due to gravity
g_w,g_v,g_θ	internal damping coefficients in flapping, lagging, and torsion, respectively
h	line of intersection of plane (x,y) of rotation and plane of cross section of blade (fig. 1)
I_1,I_2	structural moments of inertia of a blade section for bending in flapping and in lagging planes, respectively
I_{1o},I_{2o}	values of I_1 and I_2 at root of a blade
I_{max},I_{min}	mass moments of inertia per unit of blade length of a cross section about principal axes through center of gravity (here, also shear center)
I_p	mass polar moment of inertia per unit of blade length of a blade section about principal axes through center of gravity (here, also shear center)
I_{po},J_o	values of I_p and J at root of a blade

$i = \sqrt{-1}$

J	torsional moment of resistance of a blade section
j	dimensionless distance between center of gravity (here, also shear center) and aerodynamic center of a blade section (fig. 1)
K_1	dimensionless bending-stiffness parameter $\left(EI_{10}/\sigma A_o \Omega^2 l^4\right)$
K_2	dimensionless bending-stiffness parameter $\left(EI_{20}/\sigma A_o \Omega^2 l^4\right)$
k	constant in the relation $c_{d_o} = \left(c_{d_o}\right)_0 + k\alpha^2$
l	length of blade
m	number of blades in rotor system
M_1	dimensionless torsional-rigidity parameter $\left(GJ_o/\Omega^2 I_{po} l^2\right)$
$M_2 = \dfrac{\left(I_{max} - I_{min}\right)_o}{I_{po}}$	
n	subscript indicating a given principal mode of vibration
P	dimensionless parameter $\left(\pi \dfrac{\rho l c_o^3}{I_{po}}\left(\dfrac{1}{12} + d^2\right)\right)$
p	complex frequency; if $p = -R \pm i\omega$ (R and ω real) then $\omega/2\pi$ is the natural frequency in cycles per second, while $2\pi \frac{R}{\omega}$ is the logarithmic decrement
Q	dimensionless parameter $\left(\pi j \dfrac{\rho l^2 c_o^2}{I_{po}}\right)$
q	dimensionless complex frequency (p/Ω)
R	tip radius of a blade from axis of rotation
r	local radius of a blade element from axis of rotation
t	time

v	dimensionless bending (lagging) deflection in the direction of a principal axis y_1 (fig. 1) of a blade section
W	gross weight of helicopter
w	dimensionless bending (flapping) deflection in the direction of a principal axis z_1 (fig. 1) of a blade section

$$\overline{w}(t) = \int_0^1 \frac{c}{c_o} \frac{\dot{w}(\xi,t)}{\Omega} \frac{r}{R} \, d\xi$$

$$\overline{w}_1 = \int_0^1 \frac{c}{c_o} w(\xi)\frac{r}{R} \, d\xi$$

$w_{i,o}$	induced downwash velocity in steady state
w_o'/l	local slope of a deflected blade in flapping in the steady (static) state
α	local absolute angle of attack of a blade section
δ_a,δ_i	aerodynamic and internal (structural) logarithmic decrements, respectively
ϵ	eccentricity ratio (e/l)
θ	local twisting angle of a blade section

$$\overline{\theta}(t) = \int_0^1 \frac{c}{c_o} \frac{r^2}{R^2} \theta(\xi,t) \, d\xi$$

θ_o	design, or initial, blade angle (fig. 1)
θ_1	angle between zero-lift line and principal axis y_1 of a blade section (fig. 1)
ξ	dimensionless distance along a blade, measured from root
ρ	air density
σ	average density of blade material

$\tau(\xi)$ dimensionless centrifugal-force parameter $\left(\int_{\xi}^{1} \frac{A}{A_o} \frac{r}{l}\, d\xi\right)$

Ω angular speed of rotor system, radians per second

ω_n natural frequency of vibration of an elastic rotating blade in a given principal mode

ω_{cn} value of ω_n if blade had no bending stiffness, that is, if $K_1 = K_2 = 0$

ω_{en} value of ω_n if $\Omega = 0$

$' = \frac{\partial}{\partial \xi}$

$\cdot = \frac{\partial}{\partial t}$

BASIC EQUATIONS

The basic differential equations for the small flapping, lagging, and torsional vibrations, respectively, of a helicopter blade in hovering and vertical flight about its position of static equilibrium can be written in the following nondimensional form:

$$K_1\left(\frac{I_1}{I_{1o}} w''\right)'' - (\tau w')' + \frac{A}{A_o} \frac{\ddot{w}}{\Omega^2} + C \frac{r}{l} \frac{c}{c_o}\left(\frac{\dot{w}}{\Omega} - B\overline{w}\right) +$$

$$g_w K_1 \frac{\Omega}{\omega_w}\left(\frac{I_1}{I_{1o}} \frac{\dot{w}''}{\Omega}\right)'' + \frac{A}{A_o} \theta_o \ddot{v} + 2 \frac{A}{A_o} \frac{w_o'}{l} \frac{\dot{v}}{\Omega} -$$

$$C \frac{r}{l} \frac{c}{c_o}\left[\left(\theta_o + \theta_1\right) - \frac{w_{1,o}}{\Omega r}\right]\frac{\dot{v}}{\Omega} - Cd \frac{c_o}{l} \frac{r}{l}\left(\frac{c}{c_o}\right)^2 \frac{\dot{\theta}}{\Omega} -$$

$$C\left(\frac{r}{l}\right)^2 \frac{c}{c_o}\left(\theta - B \frac{l}{r} \overline{\theta}\right) = 0 \qquad (1a)$$

$$K_2\left(\frac{I_2}{I_{2o}} v''\right)'' - (\tau v')' - \frac{A}{A_o} v + \frac{A}{A_o}\frac{\ddot{v}}{\Omega^2} + C\frac{\left(c_{d_o}\right)_1}{\pi}\frac{c}{c_o}\frac{r}{l}\frac{\dot{v}}{\Omega} +$$

$$g_v K_2 \frac{\Omega}{\omega_v}\left(\frac{I_2}{I_{2o}}\frac{\dot{v}''}{\Omega}\right)'' + \frac{A}{A_o}\theta_o w - 2\frac{A}{A_o}\frac{w_o'}{l}\frac{\dot{w}}{\Omega} +$$

$$C\frac{r}{l}\frac{c}{c_o}\left(2\theta_o + \theta_1 - 2\frac{w_{i,o}}{\Omega r}\right)\left(\frac{\dot{w}}{\Omega} - B\frac{R}{l}\overline{w}\right) -$$

$$C\left(\frac{r}{l}\right)^2\frac{c}{c_o}\left[\left(2\theta_o + \theta_1 - 2\frac{w_{i,o}}{\Omega r}\right)\left(\theta - B\frac{R}{r}\overline{\theta}\right) - \frac{k}{\pi}\left(\theta_o + \theta_1\right)\theta\right] +$$

$$\frac{g}{\Omega^2 l}\frac{A}{A_o}\theta = 0 \tag{1b}$$

$$M_1\left(\frac{I_t}{I_{to}}\theta'\right)' + \left[Q\left(\frac{c}{c_o}\right)^2\left(\frac{r}{l}\right)^2 - M_2\frac{I_{max} - I_{min}}{\left(I_{max} - I_{min}\right)_o}\right]\theta -$$

$$Q\frac{R}{l}\left(\frac{c}{c_o}\right)^2\frac{r}{l}B\overline{\theta} - \frac{I_p}{I_{po}}\frac{\ddot{\theta}}{\Omega^2} - P\left(\frac{c}{c_o}\right)^3\frac{r}{l}\frac{\dot{\theta}}{\Omega} + g_\theta M_1\frac{\Omega}{\omega_\theta}\left(\frac{J}{J_o}\dot{\theta}'\right)' -$$

$$Q\left(\frac{c}{c_o}\right)^2\frac{r}{l}\left(\frac{\dot{w}}{\Omega} - B\frac{l}{R}\overline{w}\right) = 0 \tag{1c}$$

Equations (1a) and (1b) represent the equilibrium of the elastic, internal damping, centrifugal, inertia, and aerodynamic loads at any point ξ per unit length along a blade. In equation (1a) these loads, as well as the dimensionless displacement w, are in the direction of the principal axis z_1 of a cross section of a blade (see fig. 1).

In equation (1b) the loads and the displacement v are in the direction of the other principal axis y_1 of a cross section (fig. 1). If the blades are hinged then w and v represent the sum of the elastic deflections and the components of the rigid-body displacements in the directions of the principal axes z_1 and y_1, respectively.

The aerodynamic loads here have been derived, as in reference 2, by use of the three-dimensional Kutta-Joukowski theorem for quasi-stationary flow. The induced downwash, however, has now been assumed as constant throughout the rotor disk and as derivable by the simple momentum theory. (See reference 3.) The only difference, in fact, between the hovering state and the state of uniform vertical climbing or descending in equations (1a) to (1c) lies in the values of the downwash velocity $w_{i,o}$ and of the downwash parameter B.

By the momentum theory, the values of $w_{i,o}$ and B for a helicopter climbing vertically at a constant speed of v_c can be shown to be:

$$\frac{w_{i,o}}{\Omega R} = \frac{v_c}{2\Omega R} + \sqrt{\frac{1}{4}\left(\frac{v_c}{\Omega R}\right)^2 + \frac{W\left(1 + k_D\right)}{2\pi\rho R^4\Omega^2}} \tag{2a}$$

$$B = \left[\int_0^1 \frac{c}{c_o}\frac{r}{l}\,d\xi + 4\,\frac{w_{i,o}R}{\left(1 + k_D\right)mc_o l\Omega} - 2\,\frac{v_c}{\Omega c_o}\,\frac{1}{\left(1 + k_D\right)}\,\frac{1}{m}\,\frac{R^2}{l^2}\right]^{-1} \tag{2b}$$

where the drag of the helicopter in climbing has been assumed to be k_DW, and W is the gross weight of the helicopter.

It has been assumed, in the derivation of equations (1a) to (1c), that the profile-drag coefficient c_{d_o} of a blade section varies parabolically with the angle of attack.

From the condition that the static thrust must support the gross weight of the helicopter, the required value of the absolute pitch angle $\left(\theta_o + \theta_1\right)$ of a blade, assumed constant along its length, is found to be:

$$\theta_o + \theta_1 = \frac{\dfrac{W}{\pi\rho mc_o\Omega^2R^2 l} + \dfrac{w_{i,o}}{\Omega R}\displaystyle\int_0^1 \frac{c}{c_o}\frac{r}{R}\,d\xi}{\displaystyle\int_0^1 \frac{c}{c_o}\frac{r^2}{R^2}\,d\xi} \tag{3}$$

The mechanical significance of each of the terms appearing in equations (1a), (1b), and (1c) is as follows. The terms proportional to K_1, K_2, or M_1 represent the elastic resistance, while those proportional to g_w, g_v, or g_θ represent internal damping. (See reference 4.) The terms $(\tau w')'$ and $\frac{A}{A_0}\theta_0 v$ in equation (1a), the terms $(\tau v')'$, $\frac{A}{A_0} v$, and $\frac{A}{A_0}\theta_0 w$ in equation (1b), and the M_2-term in equation (1c) are due to the centrifugal loads. The terms with $\ddot{w}$, $\ddot{v}$, and $\ddot{\theta}$ represent inertia loads, while the terms $2\frac{A}{A_0}\frac{w_0'}{l}\frac{\dot{v}}{\Omega}$ and $2\frac{A}{A_0}\frac{w_0'}{l}\frac{\dot{w}}{\Omega}$ give the Coriolis effect. In equations (1a) and (1b) the terms proportional to C represent the aerodynamic loads, while in equation (1c) these loads are represented by the terms proportional to Q and to P. For mass-balanced blades, $Q = 0$. The terms in $\overline{w}$ and $\overline{\theta}$ give the effect of induced downwash. Finally the term $\frac{g}{\Omega^2 l}\frac{A}{A_0}\theta$ in equation (1b) represents the effect of the weight of a blade.

TORSIONAL VIBRATIONS

From equation (1c) it can be seen that if the blades are mass-balanced, implying $Q = 0$, then the torsional vibrations will not be coupled to the flapping vibrations. This case of balanced blades is the one which is treated in the present section on torsional vibrations and in the following two sections on vibrations in flapping and lagging.

Frequency characteristics.- By putting

$$\theta(\xi,t) = \theta(\xi)e^{pt} \tag{4}$$

into equation (1c) with $Q = 0$, and neglecting damping for the present, the following equation is obtained for $\theta(\xi)$ and $q \equiv \frac{p}{\Omega}$:

$$M_1\left(\frac{J}{J_0}\theta'\right)' - \left[M_2\frac{I_{max} - I_{min}}{\left(I_{max} - I_{min}\right)_0} + q^2\frac{I_p}{I_{po}}\right]\theta = 0 \tag{5}$$

If the blades have a uniform cross-sectional distribution and have fixed pitch at the root ($\theta = 0$ at $\xi = 0$), then the following solution is obtained for the principal torsional modes of vibration:

$$\left.\begin{aligned} \theta_n(\xi) &= \sin n \frac{\pi}{2} \xi \\ -q_n^{\,2} &= M_1 \frac{\pi^2}{4} n^2 + M_2 \approx M_1 \frac{\pi^2}{4} n^2 \end{aligned} \quad n \text{ odd, } n \geqq 1 \right\} \tag{6}$$

The term $\theta_n(\xi)$ represents the principal modes of deflection, while $\frac{\Omega}{2\pi}\sqrt{-q_n^{\,2}}$ is the natural frequency in cycles per second in the "$\frac{n+1}{2}$th" mode, if Ω is in radians per second.

It is significant to note from the results of equations (6) that, since M_2, which represents the effect of the centrifugal torque, is in actual cases negligible, the natural frequencies, as well as the modes of vibration, of a rotating helicopter blade in torsion are virtually the same as those of the same blade when it is stationary ($\Omega = 0$). This conclusion is valid for tapered, as well as for rectangular, blades (cf. appendix A).

If the blades have a variable cross-sectional distribution, then the natural frequencies of the various modes of vibration can be determined by the procedure, based on the Rayleigh-Ritz method, shown explicitly in appendix A.

Damping characteristics.- Let the natural frequency of the "kth" principal mode of vibration without damping be ω_{no}, and let the corresponding mode of deflection be $\theta_{no}(\xi)$. Then it may be assumed that for this mode with damping the deflection shape will be given approximately by $\theta_{no}(\xi)$. Therefore, with the assumption

$$\theta_n(\xi,t) = b\theta_{no}(\xi)e^{pt}$$

for the damped vibrations of the kth mode, where p may now be complex, differentiation of the integral in condition (A1) of appendix A with respect to b leads to a quadratic equation in q whose roots (with M_2 neglected) indicate the following values of the natural frequency ω_{nd}, aerodynamic logarithmic decrement δ_{an}, and internal logarithmic decrement δ_{in} of the kth mode:

$$\frac{\omega_{nd}}{\Omega} = \sqrt{\left(\frac{\omega_{no}}{\Omega}\right)^2 - \eta_{nn}^{\ 2}} \tag{7}$$

$$\delta_{an} = \frac{\pi P}{\omega_{nd}/\Omega} \frac{\int_0^1 \frac{r}{l}\left(\frac{c}{c_o}\right)^3 \theta_{no}^{\ 2}(\xi)\ d\xi}{\int_0^1 \frac{I_p}{I_{po}} \theta_{no}^{\ 2}(\xi)\ d\xi} \tag{8}$$

$$\delta_{in} = \pi g_\theta \tag{9}$$

where

$$\eta_{nn} = \frac{P}{2} \frac{\int_0^1 \frac{r}{l}\left(\frac{c}{c_o}\right)^3 \theta_{no}^{\ 2}(\xi)\ d\xi}{\int_0^1 \frac{I_p}{I_{po}} \theta_{no}^{\ 2}(\xi)\ d\xi} + \frac{g_\theta}{2} \frac{\omega_{no}}{\Omega} \tag{7a}$$

Equation (7) gives the correction, due to damping, in the natural frequencies. This correction will usually be found to be negligible in all modes above the fundamental.

Equation (8) shows that the aerodynamic logarithmic decrement in any given principal mode has an order of magnitude of $\frac{\pi P}{\omega_n/\Omega}$ and varies approximately inversely as the natural frequency. Thus, the aerodynamic logarithmic decrement decreases with the mode of vibration. Equation (9) shows that the internal logarithmic decrement is, on the other hand, independent of the natural frequency and therefore remains the same in all modes. This result for the internal damping is due to the negligible effect of the centrifugal twisting couple in any cross section of a rotating helicopter blade.

Since in general $g_\theta \ll P$, it follows from equations (8) and (9) that in the fundamental, and possibly second, mode the aerodynamic damping will be greater than the internal damping. In the high modes, however, the internal damping will predominate.

It should be noted that, although equation (9) indicates the same internal logarithmic decrement for stationary as for rotating beams, equation (8) shows that there will be no aerodynamic damping if the beam is stationary ($\delta_{an} = 0$ if $\Omega = 0$).

Numerical example.- Consider a blade of constant cross section along the span, with the data of figure 2. Then $\theta_{no}(\xi) = \sin n \frac{\pi}{2} \xi$ for the $\frac{n+1}{2}$th mode. Moreover, it is found that $M_1 = 37.0$ and $P = 7.35$. In virtually all cases, $M_2 \leqq 1$. Here let $M_2 = 0.7$. Finally let $g_\theta = 0.05$. From equations (6) it is found that (damping neglected)

$$\frac{\omega_{no}}{\Omega} = \sqrt{91.0n^2 + 0.7} \approx \sqrt{91.0n^2} = 9.58n$$

The fundamental ($n = 1$) natural frequency in torsion is thus quite high, being here almost 10 times the rotor angular speed. From equation (7a), $\eta_{nn} = 7.35\left(0.25 + \frac{0.1011}{n^2}\right) + 0.239n$. For the fundamental mode ($n = 1$), equation (7) therefore gives:

$$\frac{\omega_{1d}}{\Omega} = \sqrt{91.0 - (2.82)^2} = 9.10$$

indicating thus a decrease, due to damping, of about 5 percent in the fundamental frequency. The effect of the damping on the natural frequencies in the higher modes ($n = 3, 5, \ldots$) will be much smaller, as can be seen from equation (7) with the expressions for η_{nn} and $\frac{\omega_{no}}{\Omega}$ here in terms of n. From equation (8), it is found that $\delta_{a1} = 1.780$ and $\delta_{an} \approx \frac{1.20}{n}$ for $n \geqq 3$. Moreover, $\delta_{in} = 0.1570$ for any mode. Thus in the fundamental mode the aerodynamic damping is here over 10 times as great as the internal damping. In the second mode ($n = 3$), however, δ_a is reduced to less than one-fourth of its value in the fundamental mode, while δ_i remains the same. In the fifth and higher modes ($n \geqq 9$), the internal damping exceeds the aerodynamic damping.

BENDING VIBRATIONS IN FLAPPING

In considering bending vibrations in flapping the small coupling between the flapping and lagging vibrations will be neglected. Moreover, it will be assumed here that the blades are mass-balanced. In that case the torsional vibrations can be determined first, as in the preceding section, by obtaining θ explicitly as a function of ξ and t. This known function can then be substituted for θ into equation (1a), which is the basic equation for the motion of a blade in flapping.

Frequency characteristics.— The natural frequencies of the free vibration modes of a rotating blade in flapping are determined by the equation

$$K_1\left(\frac{I_1}{I_{1o}} w''\right)'' - (\tau w')' + \frac{A}{A_o}\frac{\ddot{w}}{\Omega^2} = 0 \tag{1a'}$$

The effect of damping is not included in equation (1a'). By putting

$$w(\xi,t) = w(\xi)e^{pt}$$

$$q \equiv \frac{p}{\Omega}$$

equation (1a') becomes:

$$K_1\left(\frac{I_1}{I_{1o}} w''\right)'' - (\tau w')' + \frac{A}{A_o} q^2 w = 0 \tag{10}$$

The values of q^2 determined by equation (10) for any principal mode of vibration can be expressed approximately in the form (see reference 5):

$$q_n^{\ 2} = q_{en}^{\ 2} + q_{cn}^{\ 2} \tag{11}$$

where $q_{en}^{\ 2}$ is the value of $q_n^{\ 2}$ when the beam is stationary $(\Omega = 0)$, and $q_{cn}^{\ 2}$ is the value of $q_n^{\ 2}$ when the blade is hypothetically rotating without bending stiffness $(K_1 = 0)$. The values of q_{en} represent the effect of the elastic resistance of the blade, while the values of q_{cn} represent the effect of the centrifugal loads.

The following remarks concerning relation (11) are pertinent. It can be shown (appendix B) that relation (11) would be exact if and only if the centrifugal loads would have no effect on the modes of deflection of a rotating beam. It can also be shown (appendix B), however, by a simple consideration of boundary conditions, that equation (11) <u>cannot</u> be an exact general statement for either fixed-ended or hinged blades and that the centrifugal loads must therefore in general have some effect on the modes of deflection. It should be further noted that the values of q_{cn} must, according to relation (11), be independent of the boundary conditions of a blade (cf. appendix in reference 5). It can be shown (appendix B), however, that there are theoretically no exact values of q_{cn}^2 for a blade fixed at the root. (This may be considered as a simple proof that relation (11) cannot be exact for fixed-ended blades.)

In spite of the fact that equation (11) cannot be an exact general statement, this relation gives sufficiently accurate results for practical purposes. This is due to the fact that, although the centrifugal loads do have an effect on the modes of deflection of a rotating beam, this effect is ordinarily small both for fixed-ended and hinged blades. (See references 6 and 7.) As a numerical check on the accuracy of relation (11), the values of the natural frequencies of a hinged blade were calculated by solving equation (10) without the use of relation (11). Approximate solutions for the lower modes (see appendix B for details of the method) showed that the natural frequencies as functions of the parameter K_1 could to a high degree of approximation be represented in the form of equation (11). Moreover, a numerical check (appendix B) was made on the effect of the root conditions of a blade on the centrifugal contributions q_{cn}^2 to the negative squared natural frequencies, and this effect was found to be practically negligible.

The values of q_{cn}^2 in relation (11) depend only on the mass <u>distribution</u> $\frac{\sigma A}{(\sigma A)_0}(\xi)$ of a blade along its length and are the same for a fixed-ended as for a hinged blade. The values of q_{en}^2, on the other hand, are in general directly proportional to the parameter K_1 of a blade and depend on the boundary conditions at the blade root, as well as on the cross-sectional distribution (i.e., plan form).

For blades of constant cross section, the values of q_{en}^2 and q_{cn}^2 in the various principal modes are as follows:

fixed-ended blades:

$$-q_{en}^2/K_1 = 12.36,\ 481,\ .\ .\ .,\ \left(\frac{\pi}{2}\,n\right)^4 \text{(approx.)},\ .\ .\ .$$

$$n \text{ odd, } n \geqq 5$$

$$-q_{cn}^2 = 1,\ 6,\ .\ .\ .,\ n(2n - 1),\ .\ .\ .$$

$$n \text{ a positive integer}$$

hinged blades:

$$-q_{en}^2/K_1 = 0,\ 236,\ .\ .\ .,\ \pi^4\left(n + \frac{1}{4}\right)^4 \text{(approx.)},\ .\ .\ .$$

$$n \text{ a positive integer}$$

$$-q_{cn}^2 = 1,\ 6,\ .\ .\ .,\ n(2n - 1),\ .\ .\ .$$

$$n \text{ a positive integer}$$

For blades of variable cross section a simple procedure, based on the Rayleigh-Ritz method, of calculating the frequency characteristics is given explicitly in appendix B.

The values of the frequency ratios ω/Ω in the various modes of vibration of a blade in flapping are evidently functions of only one parameter, K_1, in addition to being dependent on the plan form of the blade. Typical values of K_1 are usually small. For the single-tubular-spar section in figure 2, for example, $K_1 = 0.00400$. Consequently, the actual contribution of the elastic resistance of a helicopter blade to the natural flapping frequency in the fundamental mode is small in comparison with that of the centrifugal forces. In the higher modes, however, the relative importance of the elastic resistance rapidly increases. The natural frequency of a fixed-ended blade in any principal mode will, of course, be higher than that of the same blade when it is hinged. If K_1 is of the order of magnitude of unity this difference will be especially great in the lower modes. However, when $K_1 \ll 1$ (as in the tubular-spar section, fig. 2), then the natural frequencies of a fixed-ended blade are only slightly higher than those of a hinged blade. These conclusions are, in part, illustrated by figure 3 and figure 4, where numerical results are plotted for a uniform blade of low bending stiffness hinged at the root and fixed at the root, respectively.

Rigid-body oscillation as a mode of vibration.- If the blades are hinged at their roots, and if their flapping hinges pass through the axis of rotation of the rotor system, so that $\frac{r}{l} = \xi$, then the deflection shape $w = a\xi$, where a is a constant, will be an exact solution of equation (10) with $q^2 = -1$, regardless of the plan form of the blades. This shows that, if damping and small coupling effects are neglected, then a rigid-body oscillation of a blade in flapping will be an exact fundamental mode of vibration. In this mode, $\omega = \Omega$ exactly. If, however, aerodynamic damping is considered, implying addition of the term $C \frac{r}{l} \frac{c}{c_o}\left(\frac{\dot{w}}{\Omega} - B\bar{w}\right)$ to the left side of equation (1a'), then it can be seen that equation (10) will no longer be exactly satisfied by $w = a\xi$. In this case, the rigid-body oscillations will be an approximate, but not exact, mode of vibration of the blades.

Damping characteristics.- Putting $w(\xi,t) = w(\xi)e^{pt}$ into equation (1a') with the term $C \frac{r}{l} \frac{c}{c_o}\left(\frac{\dot{w}}{\Omega} - B\bar{w}\right)$ added to the left side, the following equation for $w(\xi)$ and q is obtained:

$$K_1\left(\frac{I_1}{I_{1o}} w''\right)'' - (\tau w')' + \frac{A}{A_o} q^2 w + C \frac{r}{l} \frac{c}{c_o} q\left(w - B\bar{w}_1\right) = 0$$

By mathematically transforming this equation into a stationary condition,[1] and applying the Rayleigh method with the assumption that the damping has a negligible effect on the principal modes of deflection, the following relations are obtained for the effect of the aerodynamic damping on the natural flapping frequency of any mode (denoted by subscript n) and for the aerodynamic logarithmic decrement δ_{an} in any mode in flapping:[2]

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[1]It is interesting to note that it is here possible to derive rigorously the stationary condition of an integral for a nonconservative system after making the substitution $w(\xi,t) = w(\xi)e^{pt}$.

[2]In terms of the value of the complex frequency q_{nd} for any mode with damping, equation (13) implies:

$$q_{nd} = -\left(\frac{1}{2} \gamma_{nn}/\alpha_{nn}\right) \pm i \frac{\omega_{nd}}{\Omega}$$

$$\frac{\omega_{nd}}{\Omega} = \sqrt{\left(\frac{\omega_{no}}{\Omega}\right)^2 - \left(\frac{\gamma_{nn}}{2\alpha_{nn}}\right)^2} \tag{12}$$

$$\delta_{an} = \pi \frac{\gamma_{nn}/\alpha_{nn}}{\omega_{nd}/\Omega} \tag{13}$$

where ω_{nd} is the natural frequency for the damped mode, ω_{no} is the corresponding natural frequency without damping, and γ_{nn} and α_{nn} are constants for each mode, defined as:

$$\gamma_{nn} \equiv C\left[\int_0^1 \frac{c}{c_o}\frac{r}{l} w_{no}^2(\xi)\, d\xi - B\left(\int_0^1 \frac{c}{c_o}\frac{r}{l} w_{no}(\xi)\, d\xi\right)^2\right]$$

$$\alpha_{nn} \equiv \int_0^1 \frac{A}{A_o} w_{no}^2(\xi)\, d\xi$$

Here, $w_{no}(\xi)$ is the mode of deflection without damping.

The internal logarithmic decrement, derived and discussed in detail in reference 4, is:

$$\delta_{in} = \pi g_w \frac{\omega_{en}^2}{\omega_n^2} \tag{14}$$

From equation (12), which gives the effect of the aerodynamic damping on the values of the natural frequencies, it follows that this effect decreases with the mode and will in actual cases be negligible in all principal modes with the possible exception of the fundamental. In the fundamental mode, in fact, the damping may in some cases (namely, those cases for which the parameter C is sufficiently large) be so heavy that this mode will consist of an unoscillating decaying motion instead of an oscillating motion.

Since the actual value of γ_{nn}/α_{nn} will be roughly the same (of the order of magnitude of $C/2$) for any mode, it follows from equation (13) that δ_{an} will decrease with the mode, in virtually inverse proportion to the natural frequency. Comparison of actual orders of magnitude $(O(C) = 1,\ O(g_w) = 0.05)$ shows, according to equations (13) and (14), that except in the very high modes, where $\delta_{an} + \delta_{in} \approx \delta_{in} \approx \pi g_w$, the aerodynamic damping will exceed the internal damping. The latter is, in fact, negligible in comparison with the aerodynamic damping in the fundamental mode. Because of the aerodynamic effect, the flapping vibrations in the lower modes will in general be highly damped.

Numerical example.- For a uniform blade with the data of figure 2, one finds: $K_1 = 0.00400$, $C = 1.74$, and $B = 0.532$ (equations (2a) and (2b) with $v_c = 0$ and $R = l$). For fixed-ended blades, it is permissible to substitute $w_{no}(\xi) = \xi^4 - 4\xi^3 + 6\xi^2$ into equations (12) and (13) for any mode.[3] For hinged blades, the substitution $w_{no}(\xi) = \xi$ may be made in these equations. It is thus found $\left(\text{assuming } \frac{r}{l} = \xi\right)$ that $\frac{\gamma_{nn}}{\alpha_{nn}} = \frac{2.55}{2.31} = 1.105$ when the blade is fixed-ended, and $\frac{\gamma_{nn}}{\alpha_{nn}} = 0.995$ when the blade is hinged. Moreover, for fixed-ended blades, $\frac{\omega_{1o}}{\Omega} = \sqrt{12.36 \times 0.004 + 1} = 1.024$; for hinged blades, $\frac{\omega_{1o}}{\Omega} = \sqrt{0 \times 0.004 + 1} = 1$ in the fundamental mode without damping. With aerodynamic damping taken into account, these fundamental frequencies change thus:

$$\left(\frac{\omega_{1d}}{\Omega}\right)_{\text{fixed-ended}} = \sqrt{(1.024)^2 - \left(\frac{1.105}{2}\right)^2} = 0.862$$

$$\left(\frac{\omega_{1d}}{\Omega}\right)_{\text{hinged}} = \sqrt{(1)^2 - \left(\frac{0.995}{2}\right)^2} = 0.869$$

Thus aerodynamic damping diminishes the fundamental frequency here by about 13 percent for both the fixed-ended and the hinged blade.

In the second and higher modes, however, the effect of the damping on the natural frequencies is here negligible, since the value of $\left(\omega_{no}/\Omega\right)^2$ rapidly increases (over 6 in the second mode), while the

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[3]Since the value γ_{nn}/α_{nn} does not vary greatly with the function $w_{no}(\xi)$, it follows that, in actual calculations, rough assumptions for $w_{no}(\xi)$ in equations (12) and (13) will suffice to give sufficiently accurate results for practical purposes.

value of $\left(\gamma_{nn}/2\alpha_{nn}\right)^2$ remains approximately the same, about 1/4 here. The total logarithmic decrements $(\delta_{an} + \delta_{in})$ for both the hinged and fixed-ended blades, with g_w assumed as 0.05, are shown in figures 5 and 6, respectively, plotted against the principal mode of vibration. For comparison with internal damping, the aerodynamic logarithmic decrements are also plotted separately in these figures.

Effect of torsional vibrations on flapping vibrations.- The free torsional vibrations of a mass-balanced blade in any principal mode have an effect on the free flapping vibrations of the blade similar to that which an external damped harmonic load would have, if the frequency and damping of this load are the same as the frequency and damping of the corresponding torsional mode. This effect can be determined by substituting the known function $\theta = \theta_n(\xi)e^{p_n t}$ for any principal mode in torsion (cf. TORSIONAL VIBRATIONS) into equation (1a) and then solving for $w(\xi,t)$. By this procedure it can be shown that corresponding to each principal mode of vibration in torsion of frequency ω_n and logarithmic decrement δ_n there will appear in general a component of vibration in flapping of the same frequency and logarithmic decrement. It can be shown, however, that in all actual cases the ratio of the amplitudes of these "quasi-forced" flapping vibrations to those of the corresponding "quasi-forcing" torsional vibrations will be of the order of magnitude of $\frac{1}{(\omega_n/\Omega)^2} + \frac{1}{4}\frac{c_o}{l}\frac{1}{\omega_n/\Omega}$. Since in practice $\frac{\omega_n}{\Omega} \gg 1$ (cf. TORSIONAL VIBRATIONS) it follows that the relative amplitudes of the quasi-forced flapping vibrations will be so small that the effect of free torsional vibrations on the flapping vibrations of mass-balanced blades may in practice be neglected.

BENDING VIBRATIONS IN LAGGING

The characteristics of the lagging vibrations of a helicopter blade, virtually in the plane of rotation, are discussed in this section. Coupling effects are here neglected and are discussed in the next section.

Frequency characteristics.- According to equation (1b) the natural frequencies of the free uncoupled lagging vibrations of a blade, with damping at first neglected, can be determined from the following ordinary differential equation for $v(\xi)$ and q:

$$K_2\left(\frac{I_2}{I_{2o}} v''\right)'' - (\tau v')' + \frac{A}{A_o}\left(q^2 - 1\right)v = 0 \tag{15}$$

A comparison of equation (15) with equation (10) shows that if $\frac{I_2}{I_{2o}}(\xi) \equiv \frac{I_1}{I_{1o}}(\xi)$, as will ordinarily be the case, then the following simple relation exists between the natural frequencies in lagging and flapping:

$$\omega_l^2 = \omega_f^2(K_2) - \Omega^2 \tag{16a}$$

where ω_l is the natural frequency (in cps) in lagging for any principal mode, $\omega_f(K_2)$ is the natural frequency of the corresponding mode in flapping with K_1 replaced by K_2, and Ω is the rotor angular speed (in rps here). Relation (16a) is valid for fixed-ended blades as well as for blades with <u>both</u> flapping and lagging axes hinged.

From equation (11) for flapping, it is evident that relation (16a) can also be expressed in the following form for any principal mode (characterized by subscript n):

$$\omega_{nl}^2 = \omega_{enl}^2 + \omega_{cnl}^2 \tag{16b}$$

where

$$\omega_{enl}^2 = \omega_{enf}^2 \frac{K_2}{K_1} \tag{16c}$$

$$\omega_{cnl}^2 = \omega_{cnf}^2 - \Omega^2 \tag{16d}$$

The subscripts l and f refer, respectively, to the values in lagging and in flapping. Thus the elastic contribution to the square of the lagging frequency of any principal mode is the same as the elastic contribution to the square of the corresponding flapping frequency, except that the former is proportional to K_2 while the latter is proportional to K_1. This fact is represented by figure 7. From equation (16d) it follows that the centrifugal contribution to the square of the lagging frequency of any mode is less by a fixed amount (Ω^2) than the centrifugal contribution to the square of the corresponding flapping frequency.

From equation (16a) it follows that if $K_2 = K_1$ (as in single-tubular-spar sections) then the lagging frequencies of all principal modes will be lower than the corresponding flapping frequencies. In the fundamental mode, where $\omega_f \approx \Omega$ (because of the actual low values of the bending-stiffness parameter K_1), the lagging frequency will be very low. In the higher modes, however, the difference between the flapping and lagging frequencies, if $K_1 = K_2$, becomes relatively less and less. If, on the other hand, $K_2 \gg K_1$ (as for structural airfoil sections) then the natural lagging frequencies will be greater than the corresponding flapping frequencies, often even in the fundamental mode. In that case, in fact, the following simple relation will be valid for the <u>high</u> modes: $\omega_l^2 \approx \frac{K_2}{K_1} \omega_f^2$.

It may be remarked that if $K_1 = K_2$ and $\frac{I_2}{I_{2o}}(\xi) \equiv \frac{I_1}{I_{1o}}(\xi)$, then the principal modes of deflection of the free undamped uncoupled lagging vibrations will be exactly the same as those of the corresponding flapping vibrations, provided of course that the root conditions of the blades be the same (e.g., both fixed-ended or both hinged) in both flapping and lagging.[4]

<u>Effect of a small eccentricity</u>.- If the blades are hinged at the rotor axis of rotation then in the fundamental flapping mode, $\omega_f = \Omega$. In this case, equation (16a) implies $\omega_l = 0$, signifying the absence of a centrifugal restoring force. Such a situation is easily remedied by attaching the blades at a small distance (eccentricity) e away from, instead of at, the axis of rotation. The effect of such an eccentricity, for both hinged and fixed-ended blades, can be expressed by the relation:

$$\omega^2 = \omega_{\epsilon=0}{}^2 + \epsilon \frac{N}{\alpha} \Omega^2 \tag{17}$$

where ω and Ω are in cycles per second, $\epsilon = \frac{e}{l}$,

$$N \equiv \int_0^1 X'^2(\xi) \left(\int_\xi^1 \frac{A}{A_o} d\xi \right) d\xi$$

$$\alpha \equiv \int_0^1 \frac{A}{A_o} X^2(\xi)\, d\xi$$

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[4]The condition $K_1 = K_2$ is necessary only because of the small effect of the centrifugal loads on the modes of deflection.

and $X(\xi)$ is the mode of deflection in flapping. Equation (17) is valid for any principal mode in either flapping or lagging.

In accordance with the Rayleigh method, the natural frequency in flapping ω_{cf} of a flexurally weak cable $(K_1 = 0)$ with $\epsilon = 0$ can be expressed approximately[5] in the form:

$$\left(\omega_{cf}\right)_{\epsilon=0}{}^2 = \frac{\int_0^1 \bar{\tau} X'^2 \, d\xi}{\alpha} \Omega^2$$

where

$$\bar{\tau}(\xi) = \int_1^{\xi} \frac{A}{A_0} \xi \, d\xi$$

Hence equation (17) can be written approximately in the form:

$$\omega^2 = \omega_{\epsilon=0}{}^2 + \epsilon \frac{\int_0^1 \left(\int_{\xi}^1 \frac{A}{A_0} \, d\xi\right) X'^2 \, d\xi}{\int_0^1 \left(\int_{\xi}^1 \frac{A}{A_0} \xi \, d\xi\right) X'^2 \, d\xi} \left(\omega_{cf}\right)_{\epsilon=0}{}^2 \tag{18}$$

The coefficient of $(\omega_{cf})_{\epsilon=0}{}^2$ in equation (18) is almost independent of the mode of deflection $X(\xi)$ and is greater than unity for any mode.

It is evident from equation (18) that an eccentricity of attachment of the blades increases the natural frequency of any mode in either flapping or lagging and that this increase is greater the higher the mode, since $\omega_{cf}{}^2$ increases with the mode. However, for the reason already explained, this eccentricity effect is especially important in the fundamental lagging mode of a hinged blade or of a fixed-ended blade with a low bending stiffness K_2 in the plane of rotation.

For the fundamental mode, it is permissible to substitute $X(\xi) = \xi$ for hinged blades or $X(\xi) = \xi^4 - 4\xi^3 + 6\xi^2$ for fixed-ended blades into

[5] The approximation in the equation consists in neglecting the effect of the centrifugal loads on the modes of deflection.

equation (17) or (18). If $\frac{A}{A_o}(\xi) = 1$, then for equation (17) it is thus found that $\frac{N}{\alpha} = 1.5$ for a hinged blade and $\frac{N}{\alpha} = 1.558$ for a fixed-ended blade.

Damping characteristics.- In a manner analogous to the derivation of equations (12), (13), and (14) for flapping, the following values can be obtained for the natural frequency and for the aerodynamic and the internal logarithmic decrements in any damped principal lagging mode (characterized by subscript n):

$$\frac{\omega_{nd}}{\Omega} = \sqrt{\left(\frac{\omega_{no}}{\Omega}\right)^2 - \left(\frac{\overline{\gamma}_{nn}}{2\overline{\alpha}_{nn}}\right)^2} \tag{19}$$

$$\delta_{an} = \pi \frac{\overline{\gamma}_{nn}/\overline{\alpha}_{nn}}{\omega_{nd}/\Omega} \tag{20}$$

$$\delta_{in} = \pi g_v \frac{{\omega_{en}}^2}{{\omega_n}^2} \tag{21}$$

where

$$\overline{\gamma}_{nn} \equiv \frac{C}{\pi}\left(c_{d_o}\right)_1 \int_0^1 \frac{c}{c_o} \frac{r}{l} {Y_n}^2(\xi)\ d\xi$$

$$\overline{\alpha}_{nn} \equiv \int_0^1 \frac{A}{A_o} {Y_n}^2(\xi)\ d\xi$$

and $Y_n(\xi)$ is the undamped mode of deflection in lagging. The value of the ratio $\overline{\gamma}_{nn}/\overline{\alpha}_{nn}$ is almost independent of the deflection shape $Y_n(\xi)$.

Since in actual cases the quantity $\left(\overline{\gamma}_{nn}/2\overline{\alpha}_{nn}\right)^2$ will be negligible in comparison with $(\omega_{no}/\Omega)^2$ it follows from equation (19) that the effect of damping on the lagging frequencies is in general negligible in all modes. The logarithmic decrements in lagging will vary with the mode of vibration in a manner quite similar to that in flapping (see BENDING VIBRATIONS IN FLAPPING). It must be observed, however, that the magnitude of the total (i.e., aerodynamic plus internal) damping is, in the lower modes, much smaller than that in flapping. The value of δ_{an} in lagging is, in fact, of the order of $\frac{(c_{d_o})_1}{\pi}\frac{\omega_{nf}}{\omega_{nl}}$ times that in flapping. In the fundamental mode, this ratio may be about 1/30. The internal damping in lagging is at least of the same order of magnitude as the aerodynamic damping in the fundamental mode of fixed-ended blades and is therefore relatively much more important in lagging than in flapping.

Numerical example.- For the uniform blade with the data in figure 2 and $\epsilon = 0.05$, the values of the undamped natural frequencies in lagging are plotted against the mode of vibration in figures 8 and 9 for hinged and fixed-ended blades, respectively. The logarithmic decrements in lagging with $(c_{d_o})_1 = 0.03$ and $g_v = 0.05$ are plotted in figures 10 and 11 against the modes of vibration.

Effect of torsional vibrations on lagging vibrations.- The free torsional vibrations of mass-balanced blades have an effect on the free lagging vibrations which is quite similar to that which they have on the free flapping vibrations (see BENDING VIBRATIONS IN FLAPPING). This effect, which is mathematically represented by the terms with θ in equation (1b), can, however, be shown to be negligible, because of the very low order of magnitude of the relative amplitudes of the lagging vibrations induced by the torsional vibrations.

CENTRIFUGAL, CORIOLIS, AND AERODYNAMIC COUPLING BETWEEN FLAPPING AND LAGGING

In général, there will be centrifugal, Coriolis, and aerodynamic forces which couple the flapping and lagging vibrations of a helicopter blade. These forces are represented mathematically in equations (1a) and (1b) as follows: The terms $\frac{A}{A_o}\theta_o v$ and $\frac{A}{A_o}\theta_o w$ represent the centrifugal coupling loads per unit length of the blade; the terms $2\frac{A}{A_o}\frac{w_o'}{l}\frac{\dot{v}}{\Omega}$ and $-2\frac{A}{A_o}\frac{w_o'}{l}\frac{\dot{w}}{\Omega}$ represent the Coriolis load; finally the terms $-C\frac{r}{l}\frac{c}{c_o}\left[\left(\theta_o + \theta_1\right) - \frac{w_{1,o}}{\Omega r}\right]\frac{\dot{v}}{\Omega}$

and $C\frac{r}{l}\frac{c}{c_o}\left(2\theta_o + \theta_1 - 2\frac{w_{i,o}}{\Omega r}\right)\left(\frac{\dot{w}}{\Omega} - B\frac{R}{l}\overline{w}\right)$ represent the aerodynamic coupling loads.[6] All these loads are evidently of a relatively low order of magnitude because of the actual low values of θ_o and of $\frac{w_o'}{l}$. Consequently, for a first approximation, these coupling terms may, as in the preceding two sections, be neglected in determining the frequency and damping of any principal mode in either flapping or lagging. The effects of the coupling on the frequency and damping characteristics of any mode can then be taken into account by making the corrections described in the following discussion.

Effects of coupling on frequency and damping characteristics.— By putting

$$w(\xi,t) = w(\xi)e^{pt}$$

$$v(\xi,t) = v(\xi)e^{pt}$$

$$q \equiv \frac{p}{\Omega}$$

into equations (1a) and (1b) it is possible to derive the following expressions for the complex frequencies q_{nf1} and q_{nl1} for any principal mode (characterized by subscript n) in flapping and in lagging, respectively, with the simultaneous coupling loads taken into account:

$$q_{nf1} = q_{nfo} - \frac{\frac{1}{2}q_{nfo}^2\left(4\overline{\beta}_n - \eta_n\right)\left(2\overline{\beta}_n - \eta_n\right) - \frac{1}{2}\beta_n\eta_n\theta_o q_{nfo} - \beta_n^2\theta_o^2}{\overline{\alpha}_n\left(q_{nfo}^2 - q_{nlo}^2\right)\left(2\alpha_n q_{nfo} + \gamma_n\right)} \tag{22a}$$

$$q_{nl1} = q_{nlo} - \frac{\frac{1}{2}q_{nlo}^2\left(4\overline{\beta}_n - \eta_n\right)\left(2\overline{\beta}_n - \eta_n\right) - \frac{1}{2}\beta_n\theta_o\eta_n q_{nlo} - \beta_n^2\theta_o^2}{2\overline{\alpha}_n q_{nlo}\left(q_{nlo} - q_{nfo}\right)\left[\gamma_n + \alpha_n\left(q_{nlo} + q_{nfo}\right)\right]} \tag{22b}$$

[6] It may be noted from equations (1a) and (1b) that the aerodynamic loads directly oppose the Coriolis loads but do not in general cancel the latter.

where q_{nfo} and q_{nlo} are the complex values of q in the corresponding uncoupled, though damped, modes in flapping and lagging, respectively, and where α_n, $\overline{\alpha}_n$, . . . are real constants for each uncoupled mode defined thus:

$$\alpha_n \equiv \int_0^1 \frac{A}{A_o} W_n^2 \, d\xi$$

$$\overline{\alpha}_n \equiv \int_0^1 \frac{A}{A_o} V_n^2 \, d\xi$$

$$\beta_n \equiv \int_0^1 \frac{A}{A_o} W_n V_n \, d\xi$$

$$\overline{\beta}_n \equiv \int_0^1 \frac{A}{A_o} \frac{w_o'}{l} W_n V_n \, d\xi$$

$$\eta_n \equiv C \int_0^1 \frac{r}{l} \frac{c}{c_o}\left(2\theta_o + \frac{3}{2}\theta_1 - 2\frac{w_{i,o}}{\Omega r}\right) W_n V_n \, d\xi$$

$$\gamma_n \equiv C \int_0^1 \frac{c}{c_o} \frac{r}{l} W_n^2 \, d\xi$$

Here $W_n(\xi)$ and $V_n(\xi)$ are the modes of deflection of the uncoupled flapping and lagging vibrations, respectively.

An advantageous property of equations (22a) and (22b) is that, according to the forms of these equations, the numerical values of the corrections to q_{nfo} and q_{nlo} will be rather insensitive to the forms $W_n(\xi)$ and $V_n(\xi)$ of the deflection shapes substituted into these equations. For purposes of a quick calculation it is therefore suggested that the substitutions $W_n(\xi) = V_n(\xi) = \xi$ and $W_n(\xi) = V_n(\xi) = \xi^4 - 4\xi^3 + 6\xi^2$ be made for any mode for hinged blades and for fixed-ended blades, respectively.

From equations (22a) and (22b) it is found that in general the order of magnitude of the corrections, due to coupling, in the natural frequency and logarithmic decrement of any principal mode in flapping or lagging will be that of second powers of θ_o and $\frac{w_o'}{l}$. Thus, the coupling will have what may be considered a second-order effect on the frequency and damping characteristics of any mode. This effect may, nevertheless, be quite appreciable in the fundamental lagging mode, since the damping decrement (i.e., the negative real part of q) in this mode is generally very small, while, if the bending stiffness (represented by the dimensionless parameter K_2) of the blade in lagging is small (as in the tubular-spar section, fig. 2) the natural frequency in this mode will also be low.

Numerical example.- Consider a uniform blade with the data in figure 2. Also let $\theta_1 = 0.02$, $\epsilon = 0.05$, and $g_V = g_W = 0.05$. Then from equation (3), $\theta_o = 0.1273$.

(a) Fixed-ended blades: For the fundamental mode in flapping (see Numerical example under BENDING VIBRATIONS IN FLAPPING and equation (17)),

$$q_{1fo} = -\frac{1.105}{2} + i\sqrt{(1 + 12.36 \times 0.004 + 0.05 \times 1.558) - \left(\frac{1.105}{2}\right)^2}$$

$$= -0.553 + 0.907i$$

For the fundamental mode in lagging (see figs. 9 and 11),

$$q_{1lo} = -\frac{0.178 \times 0.357}{2\pi} + 0.357i = -0.0101 + 0.357i$$

Putting $W_1(\xi) = V_1(\xi) = \xi^4 - 4\xi^3 + 6\xi^2$ and $\frac{r}{l} = \xi + \epsilon$, it is found that: $\alpha_1 = \bar{\alpha}_1 = \beta_1 = 2.31$, $\eta_1 = 0.741$, and $\gamma_1 = 3.43$.

Assuming $\frac{w_o'}{l}(\xi) = 0.203 \sin \frac{\pi}{2}\xi$,[7] it is found that $\bar{\beta}_1 = 0.434$. Substitution into equations (22a) and (22b) then yields:

$$q_{1f1} = q_{1fo} - (0.004 - 0.034i) = -0.557 + 0.941i$$

$$q_{1l1} = q_{1lo} - (-0.0048 + 0.0215i) = -0.0053 + 0.336i$$

The damping factor (real part of q) in the fundamental flapping mode is here evidently negligibly affected, while the natural frequency in this mode is increased by about 3 percent. In the fundamental lagging mode here, however, the already low damping factor is dangerously diminished (by about 50 percent), while the already low natural frequency is also diminished (by about 6 percent).

(b) Hinged blades: In this case (see Numerical example under BENDING VIBRATIONS IN FLAPPING and relation (17)),

$$q_{1fo} = -0.498 + i\sqrt{(1 + 0.05 \times 1.5) - (0.498)^2} = -0.498 + 0.909i$$

while (see figs. 8 and 10)

$$q_{1lo} = -\frac{0.143 \times 0.274}{2\pi} + 0.274i = -0.00623 + 0.274i$$

Putting $W_1(\xi) = V_1(\xi) = \xi$, it is found that: $\alpha_1 = \bar{\alpha}_1 = \beta_1 = 0.333$, $\eta_1 = 0.0775$, and $\gamma_1 = 0.464$. Moreover, assuming $\frac{w_o'}{l}(\xi) = 0.1318$, it is found that $\beta_1 = 0.0439$. Substitution into equations (22a) and (22b) then yields:

$$q_{1f1} = q_{1fo} - (0.0016 - 0.0108i) = -0.496 + 0.920i$$

$$q_{1l1} = q_{1lo} - (-0.00535 + 0.0254i) = -0.00088 + 0.249i$$

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[7] The expressions used here for $\frac{w_o'}{l}(\xi)$ are based on a first approximation for $\frac{w_o'}{l}(\xi)$ as determined by the static equation corresponding to the flapping equation (1a) with the data assumed here.

As in example (a) for fixed blades, the fundamental complex flapping frequency is here only slightly affected by the coupling, while in the fundamental lagging mode, the natural frequency and especially the damping factor are appreciably diminished.

VIBRATIONS OF UNBALANCED BLADES – FLUTTER

When the center of gravity of the cross section of a blade does not coincide with its aerodynamic center $(Q \neq 0)$ then, as can be seen from equations (1a) and (1c), the torsional vibrations will be aerodynamically coupled with the flapping vibrations. When such coupled vibrations are unstable, flutter is said to occur. In the present section simple criterions are developed for the avoidance of the flutter of rotating helicopter blades. The analysis is, however, based on quasi-stationary flow (as in the previous sections) and is therefore, strictly speaking, approximately valid only for cases of low reduced frequencies $\frac{\omega c}{2V} \approx \left(\frac{\omega}{\Omega}\right)\left(\frac{c}{R}\right)$, where V is the magnitude of the inflow velocity.

Basic equations.– By putting

$$w(\xi,t) = w(\xi)e^{pt}$$

$$\theta(\xi,t) = \theta(\xi)e^{pt}$$

$$q = \frac{p}{\Omega}$$

into equations (1a) and (1c), and neglecting second-order terms there, it is possible to combine these equations into a single stationary condition. By application of the Rayleigh method, with the assumption

$$\left.\begin{aligned} w(\xi) &= a_n X_n(\xi) \\ \theta(\xi) &= b_n Y_n(\xi) \end{aligned}\right\} \qquad (23)$$

for any given principal mode (characterized by subscript n), where $X_n(\xi)$ is the corresponding uncoupled mode of deflection in flapping while $Y_n(\xi)$ is the uncoupled mode of deflection in torsion, the following linear equations are obtained for the coefficients a_n and b_n:

$$a_n\left(J_n + q^2\alpha_n + q\gamma_n\right) + b_n C\left(\beta_n - \sigma_n\right) = 0 \tag{24a}$$

$$a_n\left(\beta_n - \sigma_n\right) - \frac{b_n}{Qq}\left(\mu_n + q^2\nu_n + \lambda_n q\right) = 0 \tag{24b}$$

where:

$$J_n \equiv K_1(1 + ig_w)\int_0^1 \frac{I_1}{I_{1o}} X_n''^2\, d\xi + \int_0^1 \tau X_n'^2\, d\xi$$

$$\alpha_n \equiv \int_0^1 \frac{A}{A_o} X_n^2\, d\xi$$

$$\lambda_n \equiv P\int_0^1 \left(\frac{c}{c_o}\right)^2\left(\frac{r}{l}\right)^2 Y_n^2\, d\xi$$

$$\gamma_n \equiv C\left[\int_0^1 \frac{c}{c_o}\frac{r}{l} X_n^2\, d\xi - B\left(\int_0^1 \frac{c}{c_o}\frac{r}{l} X_n\, d\xi\right)^2\right]$$

$$\beta_n \equiv B\left[\int_0^1 \frac{c}{c_o}\left(\frac{r}{l}\right)^2 Y_n\, d\xi\right]\left(\int_0^1 \frac{r}{l}\frac{c}{c_o} X_n\, d\xi\right)$$

$$\sigma_n \equiv \int_0^1 \left(\frac{r}{l}\right)^2 \frac{c}{c_o} X_n Y_n\, d\xi$$

$$\nu_n \equiv \int_0^1 \frac{I_p}{I_{po}}\frac{c_o}{c}\frac{r}{l} Y_n^2\, d\xi$$

$$\mu_n \equiv M_1(1 + ig_\theta)f_n\int_0^1 \frac{J}{J_o}\ Y_n'^2\ d\xi + M_2\int_0^1 \frac{I_{max} - I_{min}}{\left(I_{max} - I_{min}\right)_o}\ \frac{c_o}{c}\ \frac{r}{l}\ Y_n^2\ d\xi -$$

$$Q\left\{\int_0^1 \frac{c}{c_o}\left(\frac{r}{l}\right)^3 Y_n^2\ d\xi - B\left[\int_0^1 \frac{c}{c_o}\left(\frac{r}{l}\right)^2 Y_n\ d\xi\right]^2\right\}$$

with

$$f_n = \frac{\displaystyle\int_0^1 \frac{c_o}{c}\ \frac{r}{l}\left(\frac{J}{J_o}\ Y_n'\right)'\ d\xi}{\displaystyle\int_0^1 \left(\frac{J}{J_o}\ Y'\right)'\ d\xi}$$

The condition for the existence of a non-trivial solution (a_n and b_n not both zero) to equations (24a) and (24b) leads to the following quartic equation in the complex frequency q for the coupled mode in torsion and flapping:

$$c_4q^4 + c_3q^3 + c_2q^2 + c_1q + c_0 = 0 \tag{25}$$

where

$$c_4 = \alpha_n\nu_n$$

$$c_3 = \alpha_n\lambda_n + \gamma_n\nu_n$$

$$c_2 = J_n\nu_n + \alpha_n\mu_n + \lambda_n\gamma_n$$

$$c_1 = J_n\lambda_n + \gamma_n\mu_n + CQ(\beta_n - \sigma_n)^2$$

$$c_0 = J_n\mu_n$$

For the fundamental coupled mode, the substitutions $Y_1(\xi) = \sin\frac{\pi}{2}\xi$ and $X_1(\xi) = \xi^4 - 4\xi^3 + 6\xi^2$ (fixed-ended blade) or $X_1(\xi) = \xi$ (hinged blade) may be made for the purpose of obtaining approximate solutions for the characteristic values of q from equation (25).

Stability criterions.- Neglecting internal damping at first, it can be shown that the coupled torsional and flapping vibrations will be stable[8] if and only if

$$\mu_n > 0 \tag{26a}$$

while also

$$c_1\left(c_2c_3 - c_1c_4\right) - c_0c_3^2 > 0 \tag{26b}$$

A simple method of taking internal damping into account after conditions (26a) and (26b) have been considered is described in appendix C.

It may be noted that inequality (26a) corresponds to a necessary and sufficient condition for the stability of the uncoupled torsional vibrations of unbalanced blades, that is, for the prevention of torsional divergence.

It appears convenient in practice to regard inequalities (26a) and (26b) as conditions governing the proper chordwise location of the center of gravity of a blade section with respect to the aerodynamic center, that is, governing the suitable values of Q or j. This is in accordance with the results of references 8 and 9, where the chordwise mass distribution was found to be the chief factor determining the possibilities of flutter of helicopter blades.

If $j = 0$, that is, if the blades are mass-balanced, then flutter cannot occur, since there will be no coupling between the torsional and the flapping vibrations. If $j < 0$, then it will be found that conditions (26a) and (26b) are usually easy to satisfy, so that flutter will not occur. If, however, $j > 0$, that is, if the center of gravity of a blade section is behind the aerodynamic center, then conditions (26a) and (26b) may be violated and flutter will occur.

In this quasi-stationary type of analysis, there are two important stabilizing influences: The internal damping (proportional to g_w and g_θ) in bending and in torsion and especially the aerodynamic damping (proportional to λ_n or P) in torsion. Because of these

[8]Inequalities (26a) and (26b) are the conditions that all the real parts of q, as determined by equation (25), be negative, so that positive damping occurs.

influences, flutter may be prevented even when the center of gravity is behind, but not too far behind, the aerodynamic center of a blade section.

Numerical example.- Consider a fixed-ended uniform blade with the data in figure 2. Then, with $\frac{r}{l} = \xi$ (where $\epsilon = 0$), $X_1(\xi) = \xi^4 - 4\xi^3 + 6\xi^2$, and $Y_1(\xi) = \sin \frac{\pi}{2} \xi$ for the fundamental coupled mode, inequality (26a) leads to the condition $j < 0.186$. This is the condition for the prevention of torsional divergence (with internal damping neglected). Inequality (26b), however, leads to the condition $0.15 < j < 0.16$. Hence the critical value of j in this case for the prevention of flutter is between 0.15 and 0.16. With internal damping taken into account $(g_\theta = g_W = 0.05)$ by the method of appendix C, the critical value of j is found to be between 0.16 and 0.17. To obtain some insight into the character of the flutter vibrations in an unstable case, suppose $j = 0.17$. Then two of the roots of the complete quartic (i.e., including the internal damping terms), equation (25), are found to be $q = 0.123 \pm 3.97i$. This pair of roots evidently corresponds to an unstable mode of vibration, since the real parts are positive, indicating "negative aerodynamic damping," with a logarithmic increment of $\frac{2\pi \times 0.123}{3.97} = 0.1946$. The relative amplitudes of the flapping and torsional vibrations in this mode can be obtained by substituting the values found for q into equation (24a) or equation (24b). It is thus found that $\frac{a_1}{b_1} = -0.0214 \mp 0.0076i$, indicating that the tip amplitude of the flapping vibrations is only $3\sqrt{(0.0214)^2 + (0.0076)^2}$, or 0.067 times the tip amplitude of the torsional vibrations. This small ratio of bending amplitudes to torsional amplitudes in flutter appears to be a familiar phenomenon. (See reference 8.) The complex value of the ratio a_1/b_1 illustrates, of course, the well-known phenomenon of phase difference between the coupled bending and torsional vibrations in flutter.

Critical speed.- In addition to the design viewpoint with respect to chordwise mass distribution for the prevention of flutter for a given rotor angular speed Ω, another possible viewpoint here is that of the critical angular speed. From this point of view, all parameters, including j, may be considered as given, except the rotational speed Ω, which appears implicitly in the terms containing J_n, μ_n, β_n, and γ_n. Conditions (26a) and (26b) then determine the critical range of Ω. If $j > 0$, for example, then condition (26a) in actual cases gives an upper limit to the permissible value of Ω.

Aerodynamic coupling between lagging and torsion.[9]– It may be observed that to the order of approximation used in the analysis, there is no aerodynamic coupling between the lagging and the torsional vibrations, since the torsion equation, equation (1c), does not contain any terms in the lagging displacement v, or in derivatives of v. It may be noted, however, that with higher-order terms such a coupling would exist because of a twisting couple $c_{m_{ac}}\rho\Omega rc^2\dot{v}$, where $c_{m_{ac}}$ is the moment coefficient of a blade section about the aerodynamic center. For airfoil sections for which $c_{m_{ac}} \neq 0$, therefore, there will be coupling between the lagging and torsional vibrations. Since, however, $\dot{v}$ in equation (1c) and θ in equation (1b) will occur as products with other first-order-small quantities, these terms will be smaller than other terms appearing in these equations, indicating that the coupling between lagging and torsion is in any case small.

CONCLUSIONS

From a theoretical analysis of the frequency and damping characteristics of the free modes of vibrations of fixed-ended and hinged elastic helicopter rotor blades in hovering and vertical flight, the following conclusions are drawn:

1. The fundamental natural frequency in flapping of a rotating helicopter blade hinged at the axis of rotation is equal to the rotor angular speed. If the blade is fixed-ended at the axis of rotation, then, unless its bending stiffness is unusually high, the fundamental natural frequency in flapping will be only slightly higher than the rotor angular speed. With aerodynamic damping neglected, the rigid-body oscillations will be an exact mode of vibration in flapping of a blade hinged at the axis of rotation.

2. The natural frequency of vibration in any principal mode in flapping can be expressed approximately by means of a simple equation. In this equation the centrifugal contribution to the square of the natural frequency of a rotating blade in any mode is virtually the same whether the blade is hinged or fixed at the root. However, the elastic contribution depends on the boundary conditions at the root. A simple method of calculating the natural frequencies of a rotating beam in bending, based on these considerations, was developed.

3. A simple relation exists between the lagging and the flapping natural frequencies of any undamped uncoupled principal mode of vibration.

[9]The discussion in this paragraph is valid whether the blades are mass-balanced or not.

4. If the blades are attached at a small distance (eccentricity) away from the rotor axis of rotation, then all the natural flapping and lagging frequencies are increased. This effect is especially important in the lagging motion of hinged blades, where there would otherwise be no restoring forces in the fundamental mode.

5. The flapping vibrations are heavily damped aerodynamically in the lower modes, especially the fundamental. The internal damping in these modes is, in fact, negligible in comparison with the aerodynamic damping.

6. The aerodynamic damping in lagging is proportional to the profile-drag coefficient of a blade section and is much smaller than that in flapping. Consequently, the internal damping in the lower modes of vibration in lagging is as important as the aerodynamic damping.

7. The aerodynamic logarithmic decrements in flapping, lagging, and torsion diminish with the principal modes, in approximately inverse proportion to the vibration frequency.

8. In flapping and torsion, the importance of aerodynamic damping relative to that of internal damping diminishes with the principal mode of vibration. In lagging, however, this is not quite true in the lower modes, since for fixed-ended blades of relatively low bending stiffness in lagging the internal logarithmic decrement decreases sharply from the fundamental to the second mode.

9. The centrifugal torque in a rotating helicopter blade usually exerts a negligible effect on the natural torsional frequencies. There is considerable aerodynamic damping in torsion, due to the rotation of a blade.

10. The effect of the free torsional vibrations on the flapping and lagging vibrations of mass-balanced blades is in practice negligible, because of the relatively high natural frequencies in torsion.

11. In general, Coriolis, aerodynamic, and centrifugal coupling forces exist between the flapping and lagging motions of a rotating helicopter blade. These forces have a second-order effect on the flapping and lagging natural frequencies and damping factors, which is nevertheless appreciable in the fundamental lagging mode.

12. If the blades are unbalanced, then the resulting coupled flapping and torsional vibrations may be unstable if the cross-sectional center of gravity is too far behind the aerodynamic center. Simple criterions for prevention of such a type of flutter, based on quasi-stationary flow, were derived.

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APPENDIX A

TORSIONAL VIBRATIONS OF NONUNIFORM BLADES

Equation (5) can be mathematically transformed into the following stationary condition:

$$\delta \int_0^1 \left[M_1 \frac{J}{J_o} \theta'^2 + M_2 \frac{I_{max} - I_{min}}{\left(I_{max} - I_{min}\right)_o} + q^2 \frac{J}{J_o} \right] d\xi = 0 \tag{A1}$$

where δ is here an operator denoting variation. By assuming any mode of deflection to be expressible in the form of the series

$$\theta(\xi) = \sum_{k=1,3,5,\ldots} b_k \sin k \frac{\pi}{2} \xi \tag{A2}$$

the following set of linear homogeneous equations, written in matrix form, is obtained in the coefficients b_k:

$$\left[I_{mn} + q^2 \nu_{mn} \right] \begin{bmatrix} b_1 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} = 0 \tag{A3}$$

with $m \geqq 1$, $n \geqq 1$, and $m_{max} = n_{max}$, and where I_{mn} and ν_{mn} are constants defined thus:

$$I_{mn} \equiv M_1 \int_0^1 \frac{J}{J_0} mn \frac{\pi^2}{4} \cos \frac{m\pi}{2} \xi \cos \frac{n\pi}{2} \xi \, d\xi +$$

$$M_2 \int_0^1 \frac{I_{max} - I_{min}}{\left(I_{max} - I_{min}\right)_0} \sin \frac{m\pi}{2} \xi \sin \frac{n\pi}{2} \xi \, d\xi$$

$$\nu_{mn} \equiv \int_0^1 \frac{J}{J_0} \sin \frac{m\pi}{2} \xi \sin \frac{n\pi}{2} \xi \, d\xi$$

It will be found in actual cases that the M_2-term in the expression for I_{mn} is negligible in comparison with the M_1-term.

Equations (A3), written out explicitly, look thus:

$$b_1\left(I_{11} + q^2\nu_{11}\right) + b_3\left(I_{31} + q^2\nu_{31}\right) + \ldots + b_n\left(I_{n1} + q^2\nu_{n1}\right) + \ldots = 0$$

$$b_1\left(I_{13} + q^2\nu_{13}\right) + b_3\left(I_{33} + q^2\nu_{33}\right) + \ldots + b_n\left(I_{n3} + q^2\nu_{n3}\right) + \ldots = 0$$

$$b_1\left(I_{1n} + q^2\nu_{1n}\right) + b_3\left(I_{3n} + q^2\nu_{3n}\right) + \ldots + b_n\left(I_{nn} + q^2\nu_{nn}\right) + \ldots = 0$$

The condition for the existence of a non-trivial solution to equations (A3) is that the determinant of the coefficients of these equations vanish. This leads to the following determinant equation:

$$\left| I_{mn} + q^2\nu_{mn} \right| = 0 \qquad \text{(A4)}$$

Equation (A4) determines the natural frequencies of the various principal modes. Although equation (A4) is, for <u>exact</u> calculations, of infinite degree with an infinite number of terms, it will be found in actual calculations that this equation is rapidly convergent; that is, that it suffices to use only a few terms in series (A2) to obtain to a <u>sufficient approximation</u> the values of the natural frequencies of the lower modes. This convergent property is one of the chief advantages of the Rayleigh-Ritz method outlined here.

APPENDIX B

MATHEMATICAL DETAILS FOR BENDING VIBRATIONS IN FLAPPING

(a) Proof that relation (11) would be exact if and only if the centrifugal loads would have no effect on the modes of deflection of a rotating beam:

Suppose, first, that relation (11) is an exact general relation; that is, suppose that the relation

$$q_n^2 = q_{cn}^2 + q_{en}^2 \tag{B1}$$

is in general exactly valid. By definition of q_{cn} and q_{en} it follows from equation (10) that

$$\left.\begin{aligned} q_{cn}^2 &= \frac{(\tau w_{cn}')'}{\frac{A}{A_o} w_{cn}} \\ q_{en}^2 &= -\frac{K_1\left(\frac{I_1}{I_{1o}} w_{en}''\right)''}{\frac{A}{A_o} w_{en}} \end{aligned}\right\} \tag{B2}$$

where $w_{cn}(\xi)$ is a principal mode (characterized by subscript n) of deflection of a rotating cable without bending stiffness, while $w_{en}(\xi)$ is the corresponding mode of deflection of a nonrotating beam. By putting equations (B1) and (B2) into equation (10), the following equation is obtained:

$$K_1\left[\left(\frac{I_1}{I_{1o}} w_n''\right)'' - \frac{w_n}{w_{en}}\left(\frac{I_1}{I_{1o}} w_{en}''\right)''\right] + \left[\frac{w_n}{w_{cn}}(\tau w_{cn}')' - (\tau w_n')'\right] = 0 \tag{B3}$$

where $w_n(\xi)$ is the mode of deflection of a rotating beam with bending stiffness.

In order that relation (B1) be exact regardless of the value of K_1 it is necessary that equation (B3) be satisfied for any value of K_1, and since $w_{en}(\xi)$ and $w_{cn}(\xi)$ are independent of K_1, it is therefore necessary that the following two equations be satisfied:

$$\left(\frac{I_1}{I_{1o}} w_n''\right)'' - \frac{w_n}{w_{en}}\left(\frac{I_1}{I_{1o}} w_{en}''\right)'' = 0 \tag{B4}$$

$$\frac{w_n}{w_{cn}}\left(\tau w_{cn}'\right)' - \left(\tau w_n'\right)' = 0 \tag{B5}$$

The unique solution of equation (B4) for $w_n(\xi)$ satisfying all of the boundary conditions is

$$w_n(\xi) = w_{en}(\xi) \tag{B6}$$

Similarly the solution of equation (B5) for $w_n(\xi)$ is

$$w_n(\xi) = w_{cn}(\xi) \tag{B7}$$

Equation (B6) proves that relation (B1) could be exact only if the modes of deflection of a rotating beam were exactly the same as those of the same beam when it is stationary.

To prove the converse, suppose relation (B6) were in general exactly valid. Then substitution of relation (B6), together with equation (B2), into equation (10) would lead to the following relation:

$$\frac{A}{A_o}\left[\left(q_n^2 - q_{en}^2\right)w_{en}(\xi) - q_{cn}^2 w_{cn}(\xi)\right] + \left[\left(\tau w_{cn}'\right)' - \left(\tau w_{en}'\right)'\right] = 0 \tag{B8}$$

Observing that $w_{en}(\xi)$ and $w_{cn}(\xi)$ are functions not containing q_{en}^2 it follows from equation (B8) that $w_{cn}(\xi) = w_{en}(\xi)$, while also $q_n^2 = q_{en}^2 + q_{cn}^2$. Thus relation (B6) would imply relation (11).

(b) Proof that relation (11) cannot be an exact general statement, whether the blades are hinged or fixed-ended:

It has been shown that relation (11) implies relation (B7). It will be shown, however, that relation (B7) cannot be an exact general statement because $w_{cn}(\xi)$ cannot in all cases satisfy the boundary conditions $w_{cn}'' = w_{cn}''' = 0$ required at the free tip of a beam with bending stiffness.

By definition, $w_{cn}(\xi)$ satisfies the equation (cf. equation (10)):

$$-\left(\tau w_{cn}'\right)' + \frac{A}{A_o} q_{cn}^2 w_{cn} = 0 \tag{B9}$$

Differentiating equation (B9) three times with respect to ξ, noting that $\tau' = -\frac{A}{A_o}\xi$ $\left(\text{for simplicity, it is assumed here that } \frac{r}{l} = \xi\right)$, and assuming that $w_{cn}'' = w_{cn}''' = 0$ at $\xi = 1$, the following relation is obtained, where w_1, $\left(\frac{A}{A_o}\right)_1$, and derivatives thereof denote the values of $w_{cn}(\xi)$, of $\frac{A}{A_o}(\xi)$, and of their derivatives at $\xi = 1$:

$$3\left(q_{cn}^2 + 1\right)\left(\frac{A}{A_o}\right)_1'' w_1' + \left(\frac{A}{A_o}\right)''' w_1' + q_{cn}^2\left(\frac{A}{A_o}\right)''' w_1 - 4\left(\frac{A}{A_o}\right)_1 w_1^{iv} = 0 \tag{B10}$$

Since $w_{cn}(\xi) = w_{en}(\xi)$, it follows from equation (B2) that

$$w_1^{iv} = -\left(\frac{A}{A_o}\right)_1 q_{en}^2\left(\frac{I_{1o}}{I_1}\right)_1 \frac{w_1}{K_1} \tag{B11}$$

Moreover, equation (B9) implies

$$w_1' = -q_{cn}^2 w_1 \tag{B12}$$

By putting equations (B11) and (B12) into equation (B10), the following relation, required to be valid for any type of blade in every principal mode, is obtained:

$$4\left(\frac{A}{A_o}\right)_1^2 q_{en}^2 = 3\left(q_{cn}^2 + 1\right)q_{cn}^2\left(\frac{A}{A_o}\right)_1'' K_1\left(\frac{I_1}{I_{1o}}\right)_1 \tag{B13}$$

It is evident that relation (B13) cannot be valid <u>in general</u>[10] since, for example, in the case $\left(\frac{A}{A_o}\right)_1'' \geqq 0$ equation (B13) leads to an absurdity, the left side being negative while the right side is positive or zero.

(c) Proof that there are no exact values of q_{cn}^2 for a fixed-ended blade:

If there were exact values of q_{cn}^2 then these would satisfy equation (B9), with the appropriate functions $w_{cn}(\xi)$. However, for fixed-ended blades, $w_{cn}(0) = w_{cn}'(0) = 0$. Hence from equation (B9) it follows that $w_{cn}(\xi)$ would have to satisfy also the condition tion $w_{cn}''(0) = 0$. Moreover, by successive differentiation of equation (B9), it is seen that all of the derivatives of $w_{cn}(\xi)$ would have to vanish at the root of the blade. Hence $w_{cn}(\xi) \equiv 0$. This indicates that there cannot be any exact values of q_{cn}^2 with corresponding functions $w_{cn}(\xi)$ satisfying equation (B9) for a fixed-ended blade.

(d) Numerical check on accuracy of approximation of relation (11):

Equation (10) can be mathematically transformed into the following stationary condition:

$$\delta\int_0^1 \left[\frac{1}{2} K_1 \frac{I_1}{I_{1o}} w''^2 + \frac{1}{2}\tau w'^2 + \frac{1}{2}\frac{A}{A_o} q^2 w^2\right] d\xi = 0 \tag{B14}$$

The Rayleigh-Ritz method can be applied to condition (B14) by setting

$$w = \Sigma a_m X_m(\xi) \tag{B15}$$

where $X_m(\xi)$ are given functions satisfying the boundary conditions.

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[10]There may be special cases in which equation (B13) is valid. For example equation (B13) may, for a given blade, be valid in one particular mode, as in the fundamental mode of a hinged blade, where $q_{e1}^2 = 0$ while $q_{c1}^2 = -1$.

The most general sets of polynominals independently satisfying the boundary conditions for fixed-ended and for hinged blades are:

$$\left.\begin{aligned}
&\text{fixed-ended:}\\
&\quad X_m(\xi) = \xi^m - \frac{1}{6}\,m(m-1)(m-2)\xi^3 + \frac{1}{2}\,m(m-1)(m-3)\xi^2\\
&\qquad\qquad m \geqq 4\\
&\text{hinged:}\\
&\quad X_m(\xi) = \xi^m + \frac{1}{6}\,m(m-1)(m-4)\xi^3 - \frac{1}{12}\,m(m-1)(m-3)\xi^4\\
&\qquad\qquad m = 1, 5, 6, 7, \ldots
\end{aligned}\right\} \qquad \text{(B16)}$$

The Rayleigh-Ritz method (see, for example, reference 10) was applied here for fixed-ended blades with the first three terms of equations (B16) in series (B15). By substituting the series for w into the integrand of condition (B14) and differentiating this integrand with respect to each of the three coefficients a_m, a set of three linear homogeneous equations in a_m was obtained, the vanishing of whose determinant led to a cubic equation in the negative squared natural frequencies q^2. By investigating the solutions of this cubic, it was found that all the three roots could, to a high degree of approximation, be expressed in the form $q_n^2 = -f_{en}K_1 - f_{cn}$ where f_{en} and f_{cn} were positive numbers, independent of K_1. This verified relation (11), since the quantity $-f_{en}K_1$ could then be interpreted as q_{en}^2, while the quantity $-f_{cn}$ could be interpreted as q_{cn}^2.

(e) Effect of root conditions on centrifugal contributions to natural frequencies:

As can be seen from reference 5 (appendix), the values of q_{cn}^2 in relation (11) are independent of whether a blade is hinged or fixed at the root, since the only boundary condition which can in an exact solution, by Bessel series, be satisfied by equation (B9) for $w_{cn}(\xi)$ is $w_{cn}(0) = 0$. It is significant to check whether the approximate method outlined in item (d) leads to the same results in actual calculations. For this purpose the stationary condition, equation (B14), with $K_1 = 0$ was applied by using four terms in series (B15) with $X_m(\xi)$ for fixed-ended blades and then with $X_m(\xi)$ for hinged blades (see equations (B16)). For fixed-ended blades the following values of q_{cn}^2 were obtained for the first two principal modes: $q_{c1}^2 = -1.035$ and $q_{c2}^2 = -6.21$. For hinged blades, the corresponding values obtained were $q_{c1}^2 = -1$ and $q_{c2}^2 = -6.05$. Thus the values of q_{cn} $(n = 1,2)$ obtained by using series (B16) for fixed-ended blades were within 2 percent of those obtained by using the series for hinged blades.

As a further numerical check the Rayleigh-Ritz method was applied to condition (B14) with $K_1 = 0$ with the first four terms in series (B15) with

$$\left.\begin{aligned} X_m(\xi) &= \xi^m \\ m &\geqq 1 \end{aligned}\right\} \tag{B17}$$

This series satisfies only the single boundary condition of zero deflection at the root. The roots obtained for q_{cn} in the first two modes were: $q_{c1}^2 = -1$ and $q_{c2}^2 = -6$. Thus the simple series, equation (B17), led to nearly the same results as either of series (B15). It can, in fact, be shown that for a constant or linearly varying cross-sectional area the values of q_{cn}^2 thus obtained in any mode will be the same as the exact values obtained by Bessel series, provided a sufficient number of terms is used in series (B17).

(f) Method of calculating natural frequencies of any mode for rotating blade of variable cross section:

By making use of relation (11) and of the fact that the centrifugal contributions q_{cn}^2 to the negative squared natural frequencies are negligibly affected by the root conditions of a blade, the Rayleigh-Ritz method applied to condition (B14) leads to the following approximate method of calculating the elastic, negative squared frequencies q_{en}^2. The values of q_{en}^2/K_1 are the roots for q^2 of the determinantal equation:[11]

[11]The determinant in equation (B18) is obviously diagonally symmetric.

$$\begin{vmatrix} J_{11}+\alpha_{11}q^2 & J_{15}+\alpha_{15}q^2 & J_{16}+\alpha_{16}q^2 & \cdots & J_{1k}+\alpha_{1k}q^2 \\ J_{51}+\alpha_{51}q^2 & J_{55}+\alpha_{55}q^2 & J_{56}+\alpha_{56}q^2 & \cdots & J_{5k}+\alpha_{5k}q^2 \\ J_{61}+\alpha_{61}q^2 & J_{65}+\alpha_{65}q^2 & J_{66}+\alpha_{66}q^2 & \cdots & J_{6k}+\alpha_{6k}q^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ J_{k1}+\alpha_{k1}q^2 & J_{k5}+\alpha_{k5}q^2 & J_{k6}+\alpha_{k6}q^2 & \cdots & J_{kk}+\alpha_{kk}q^2 \end{vmatrix} = 0 \qquad \text{(B18)}$$

where

$$J_{hi} \equiv \int_0^1 \frac{I_1}{I_{1o}}(\xi)X_h''(\xi)X_i''(\xi)\,d\xi$$

$$\alpha_{hi} \equiv \int_0^1 \frac{A}{A_o}(\xi)X_h(\xi)X_i(\xi)\,d\xi$$

and $X_h(\xi)$ or $X_i(\xi)$ is given by series (B16), which satisfies all of the boundary conditions in bending.[12]

The values of q_{cn}^2 are the roots for q^2 of the determinantal equation:

$$\begin{vmatrix} T_{11}+\alpha_{11}q^2 & T_{12}+\alpha_{12}q^2 & T_{13}+\alpha_{13}q^2 & \cdots & T_{1l}+\alpha_{1l}q^2 \\ T_{21}+\alpha_{21}q^2 & T_{22}+\alpha_{22}q^2 & T_{23}+\alpha_{23}q^2 & \cdots & T_{2l}+\alpha_{2l}q^2 \\ T_{31}+\alpha_{31}q^2 & T_{32}+\alpha_{32}q^2 & T_{33}+\alpha_{33}q^2 & \cdots & T_{3l}+\alpha_{3l}q^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{l1}+\alpha_{l1}q^2 & T_{l2}+\alpha_{l2}q^2 & T_{l3}+\alpha_{l3}q^2 & \cdots & T_{ll}+\alpha_{ll}q^2 \end{vmatrix} = 0 \qquad \text{(B19)}$$

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[12] If the blades are fixed-ended instead of hinged at their roots, then, according to series (B16), the subscript 1 in the determinant in equation (B18) should be replaced by the subscript 4.

where

$$T_{fg} \equiv \int_0^1 \tau(\xi) X_f'(\xi) X_g'(\xi)\ d\xi$$

with

$$\tau(\xi) \equiv \int_\xi^1 \frac{A}{A_0}(\xi)\ \frac{r}{l}\ d\xi$$

and $X_f(\xi)$ or $X_g(\xi)$ is now given by the simple series, equation (B17), which satisfies only the condition of zero deflection at the root. In equation (B18), $(k - 3)$ is the number of terms used in series (B16), while in equation (B19), l is the number of terms used in series (B17).

The chief advantage of the method outlined is that the convergence is rapid, even for modes above the fundamental. Consequently, only relatively few terms need be used in actual cases in series (B15) to obtain the natural frequencies to a sufficient approximation. Suppose, for example, it is desired to obtain the natural frequencies for the three lowest modes. Then one first chooses $k = 3$ and $l = 3$. The three values of q^2 obtained from equations (B18) and (B19) represent the values corresponding to the three lowest modes. The two lowest values of q^2 will usually be more accurate than the third. To improve the accuracy one then chooses $k = 4$ and $l = 4$ and obtains again the roots for q^2 corresponding to the first three modes, as well as an additional root corresponding to the fourth mode. It will usually be found that at least the two lowest roots for q^2 thus obtained are negligibly different from the two lowest roots obtained with $k = 3$ and $l = 3$. This means that the natural frequencies of at least the two lower modes have been determined with sufficient accuracy. The natural frequency of the third mode could be obtained to a still further approximation by taking $k = 5$ and $l = 5$. In general it can be stated that, if for $k = \bar{k}$ (say) and for $k = \bar{k} + 1$ the s lowest roots for q^2 of equation (B18) are practically the same, then these are the values of $\frac{q_{en}^2}{K_1}$ for the s lowest modes. The method is similarly valid for q_{cn}^2.

If the order of the determinant in either equation (B18) or (B19) is higher than three, then experience has indicated that a convenient method of solving the determinantal equation for any root is by systematic trial and error. One chooses two values of q^2 which are believed to be fairly close to an actual root and evaluates the determinant for each of these values. The value of q^2 then obtained by linear interpolation (preferably) or extrapolation, according to which the determinant would be zero, will then be a closer approximation to the actual root.

A convenient method of evaluating a determinant of any order is to transform it into a triangular form. This method will be illustrated here for a third-order determinant. Consider the determinant

$$D = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

It is desired to obtain a determinant equal to D where the elements in the present position of a_{21}, a_{31}, and a_{32} will be zero. Multiply the first column by a_{21}/a_{11} and then subtract this column from the second column. Also, multiply the first column by a_{31}/a_{11} and subtract it from the third. Then a determinant of the following form is obtained:

$$D = D' = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{12} & b_{22} & b_{32} \\ a_{13} & b_{23} & b_{33} \end{vmatrix}$$

In D' now multiply the second column by b_{32}/b_{22} and subtract it from the third column. Then the following result is obtained:

$$D = D' = D'' = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{12} & b_{22} & 0 \\ a_{13} & b_{23} & c_{33} \end{vmatrix} = a_{11} \cdot b_{22} \cdot c_{33}$$

These calculations should be performed with a sufficient number of significant figures to avoid any large errors due to relatively small differences.

APPENDIX C

INTERNAL DAMPING IN UNBALANCED BLADES

The characteristic equation, equation (25), including internal damping can be written in the form:

$$F_1(q) + iF_2(q) = 0 \tag{C1}$$

where F_1 and F_2 are polynominals in q with real coefficients. The effect of internal damping is represented by F_2. Since $\left|F_2(q)\right| \ll \left|F_1(q)\right|$, equation (C1) can be solved for q to a good approximation by applying Newton's method as follows. The equation

$$F_1(q) = 0 \tag{C2}$$

can be solved first. Suppose one of the roots thus obtained is q_o. Then a corrected value q_1 of q_o due to the internal damping will be

$$q_1 = q_o - \frac{iF_2(q_o)}{F_1'(q_o) + iF_2'(q_o)} \tag{C3}$$

Equation (C3) can be used to determine the critical value of j thus. Suppose that from condition (26b) it is found that to prevent flutter it is necessary that $j < j_o$ (say). Then j_o is the critical value of j with internal damping neglected. To take internal damping into account, slightly higher values of j may be chosen, and the corresponding roots of the quartic, equation (25), including internal damping can then be obtained by means of equations (C2) and (C3). The minimum value of j for which at least one root of the complete quartic, equation (25), will have a positive real part will be the critical value of j with internal damping.

Near the critical value of j, the imaginary part of q_o will generally be much greater than the real part, so that the actual computations can be simplified by putting $q_o = i\omega$ (ω real) in equation (C3).

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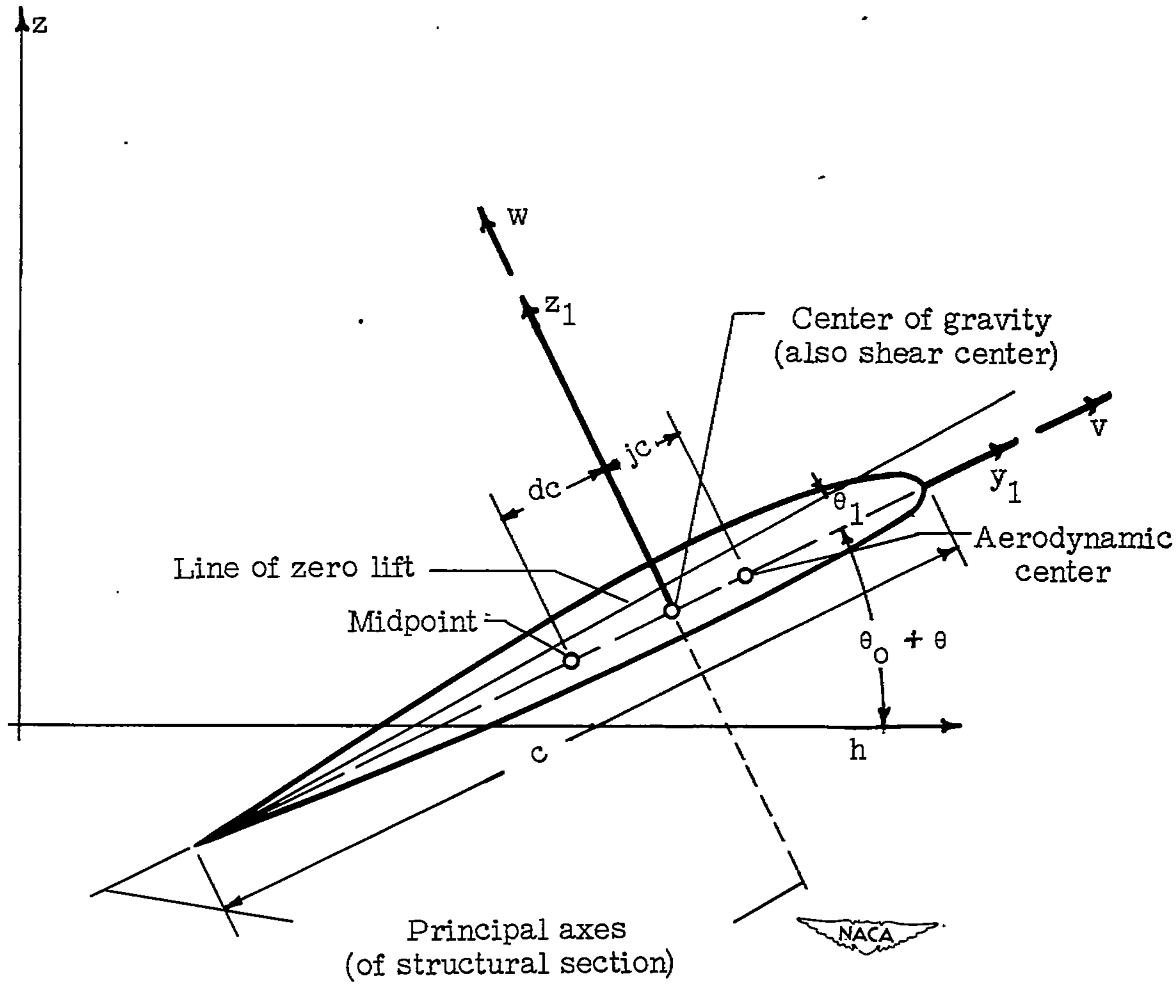
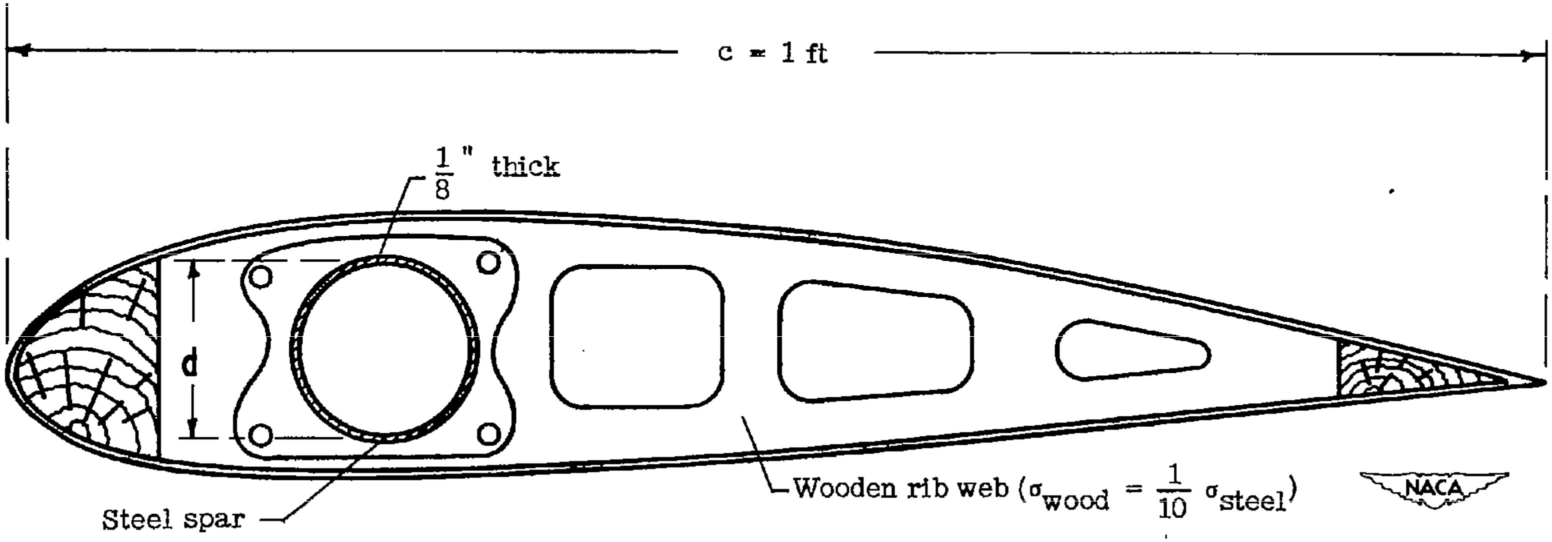


Figure 1.- Cross section of a blade.



Ω = 25 radians/sec = 239 rpm
R = 20 ft
W = 3000 lb
Three blades
d/c = 1/8

Figure 2.- Blade section used in numerical examples.

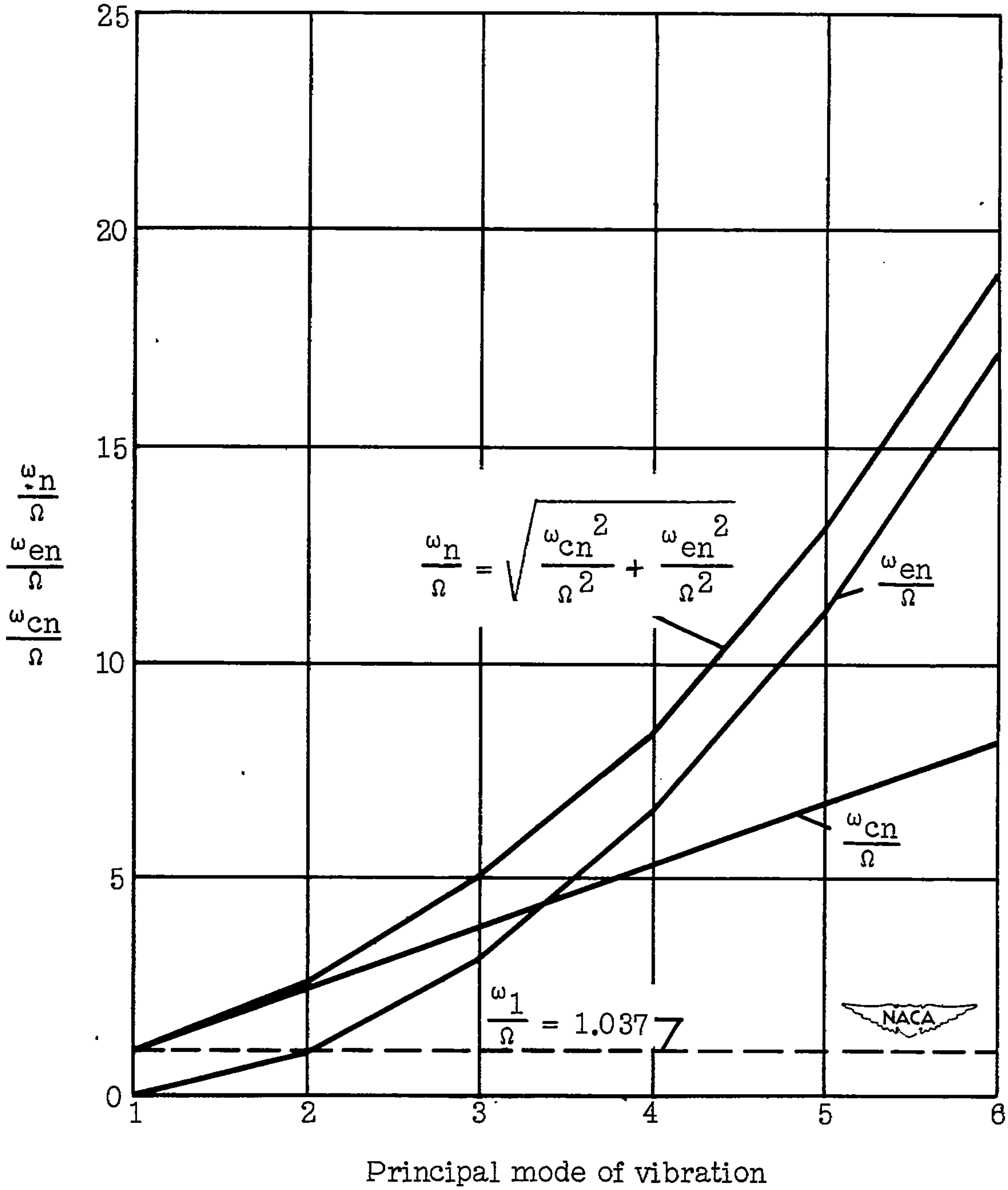


Figure 3.- Flapping frequencies of hinged blade. Constant cross section; $K_1 = 0.00400$; $\epsilon = 0.05$.

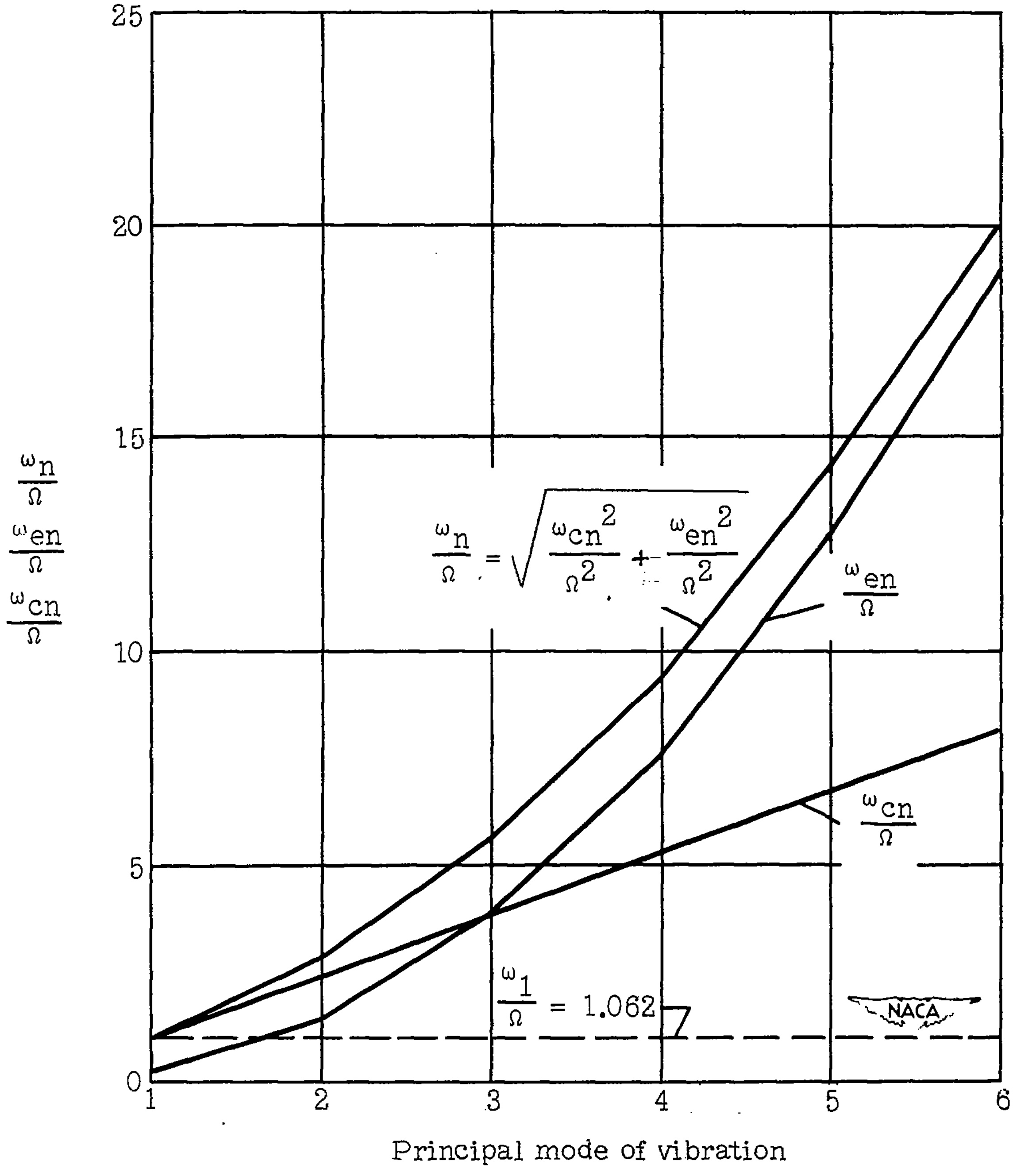


Figure 4.- Flapping frequencies of fixed-ended blade. Constant cross section; $K_1 = 0.00400$; $\epsilon = 0.05$.

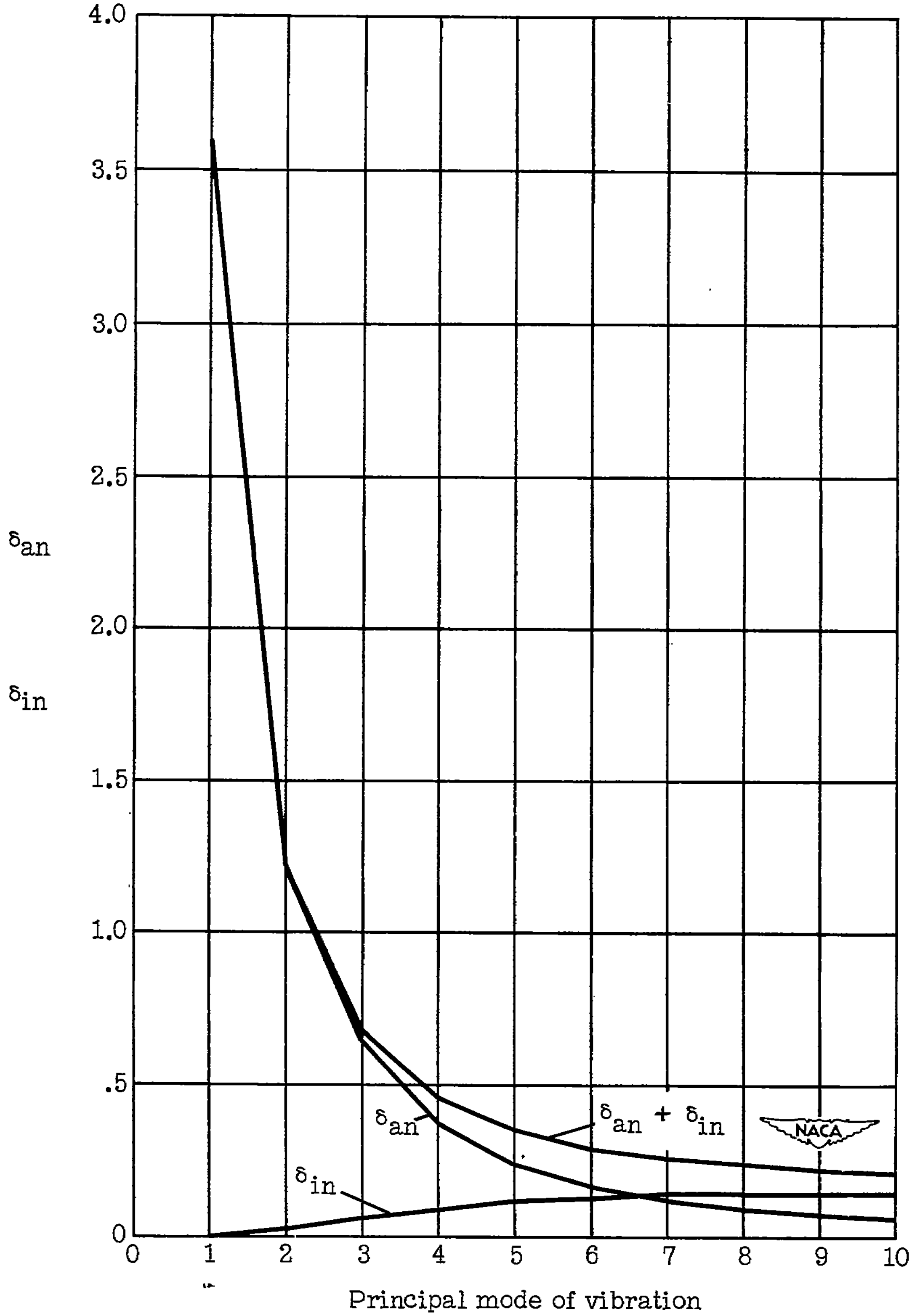


Figure 5.- Aerodynamic and internal logarithmic decrements of hinged blade in flapping. Constant cross section; $K_1 = 0.00400$; $\epsilon = 0.05$.

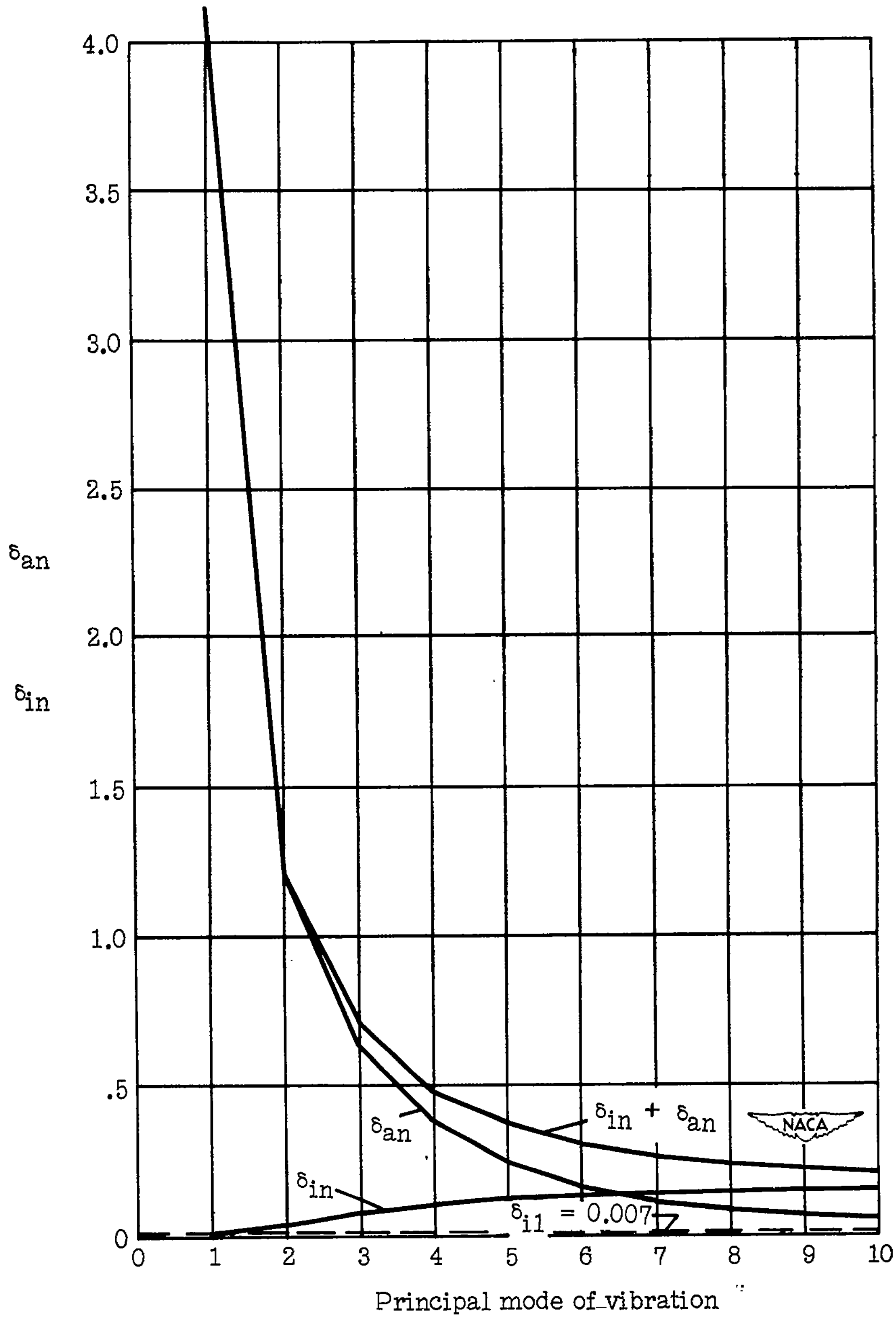


Figure 6.- Aerodynamic and internal logarithmic decrements of fixed-ended blade in flapping. Constant cross section; $K_1 = 0.00400$; $\epsilon = 0.05$.

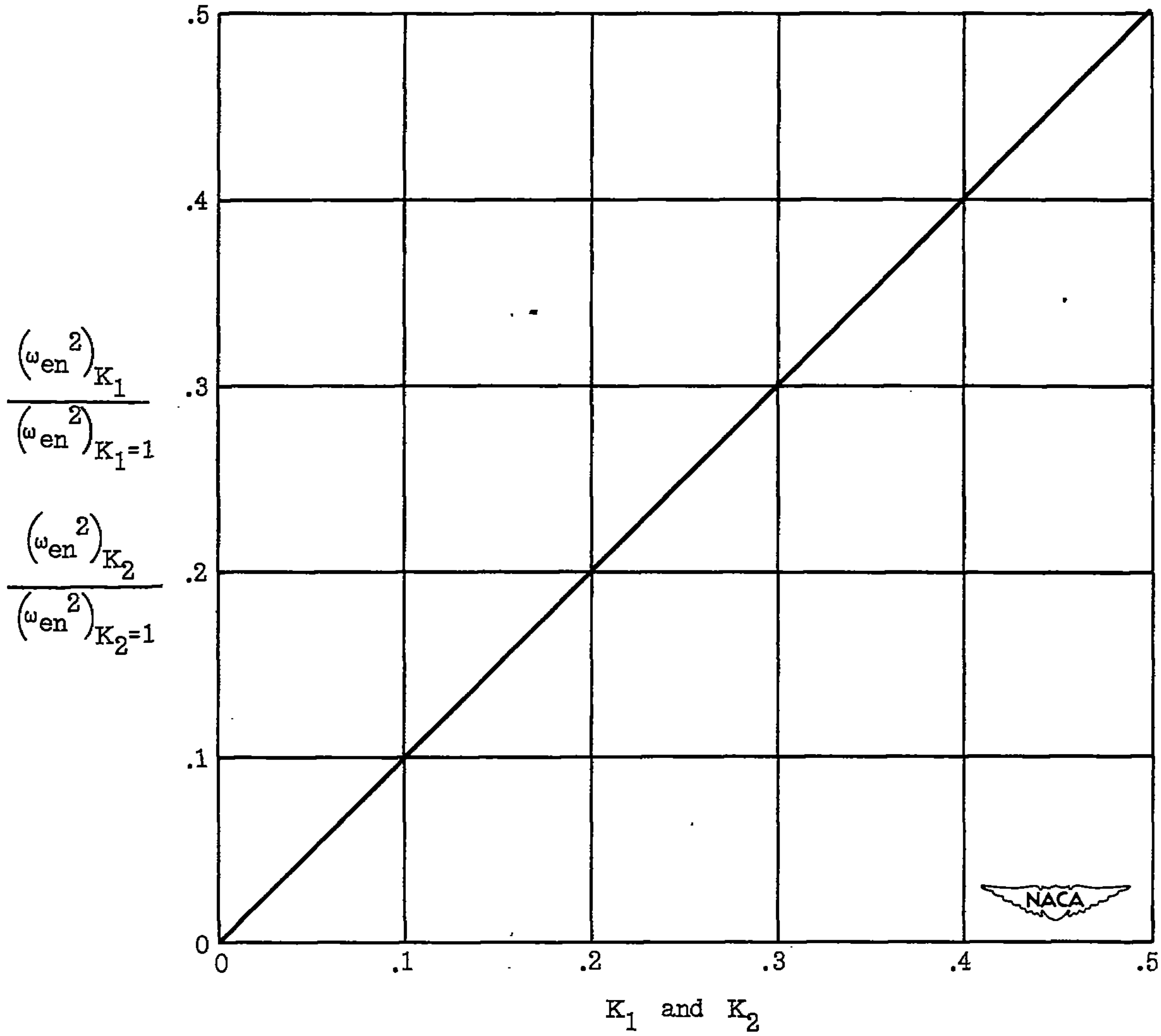


Figure 7.- Variation of ω_{en}^{2} with K_1 (flapping) or K_2 (lagging) for any mode (n).

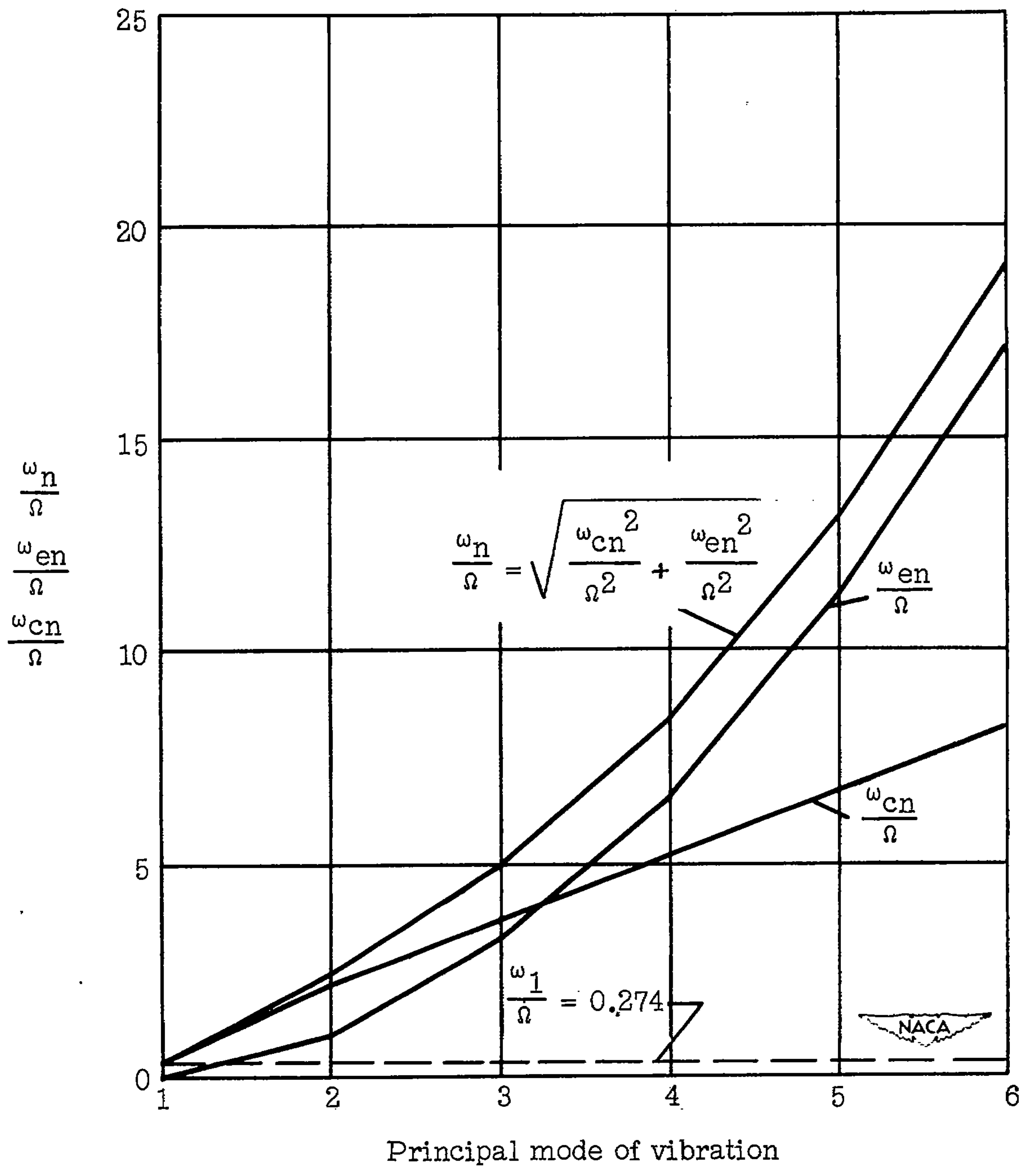


Figure 8.- Lagging frequencies of hinged blade. Constant cross section; $K_2 = 0.00400$; $\epsilon = 0.05$.

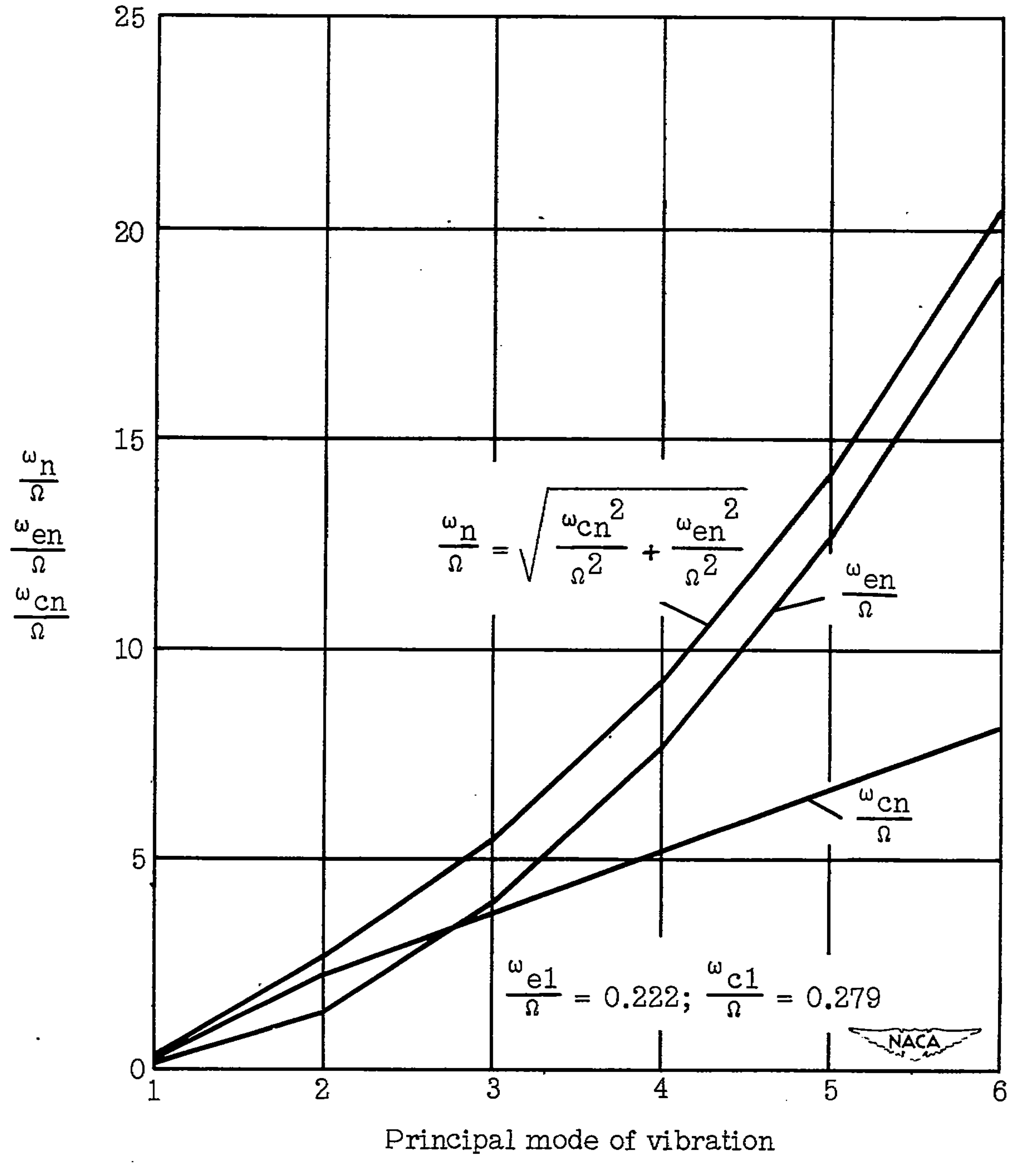


Figure 9.- Lagging frequencies of fixed-ended blade. Constant cross section; $K_2 = 0.00400$; $\epsilon = 0.05$.

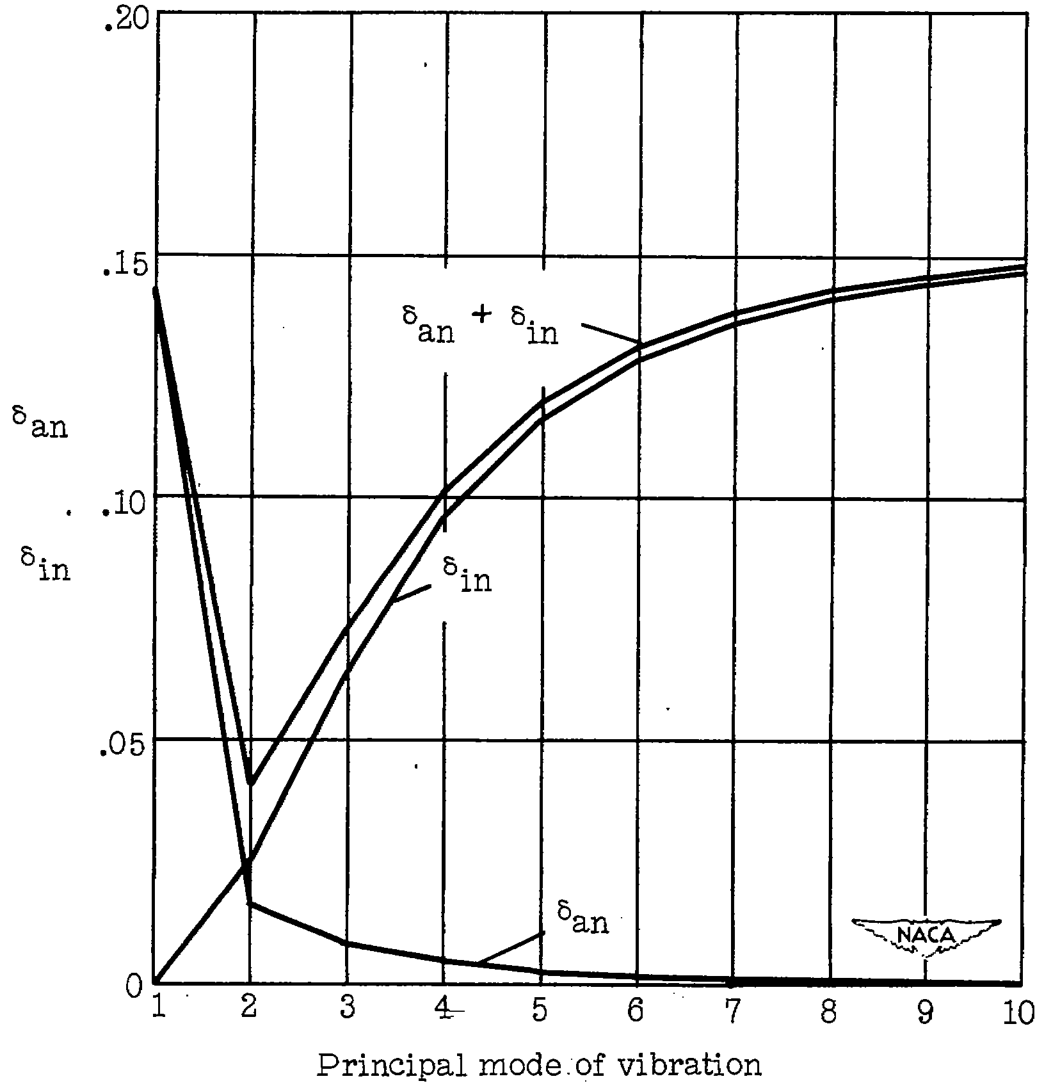


Figure 10.- Aerodynamic and internal logarithmic decrements of hinged blade in lagging. Constant cross section; $K_2 = 0.00400$; $\epsilon = 0.05$.

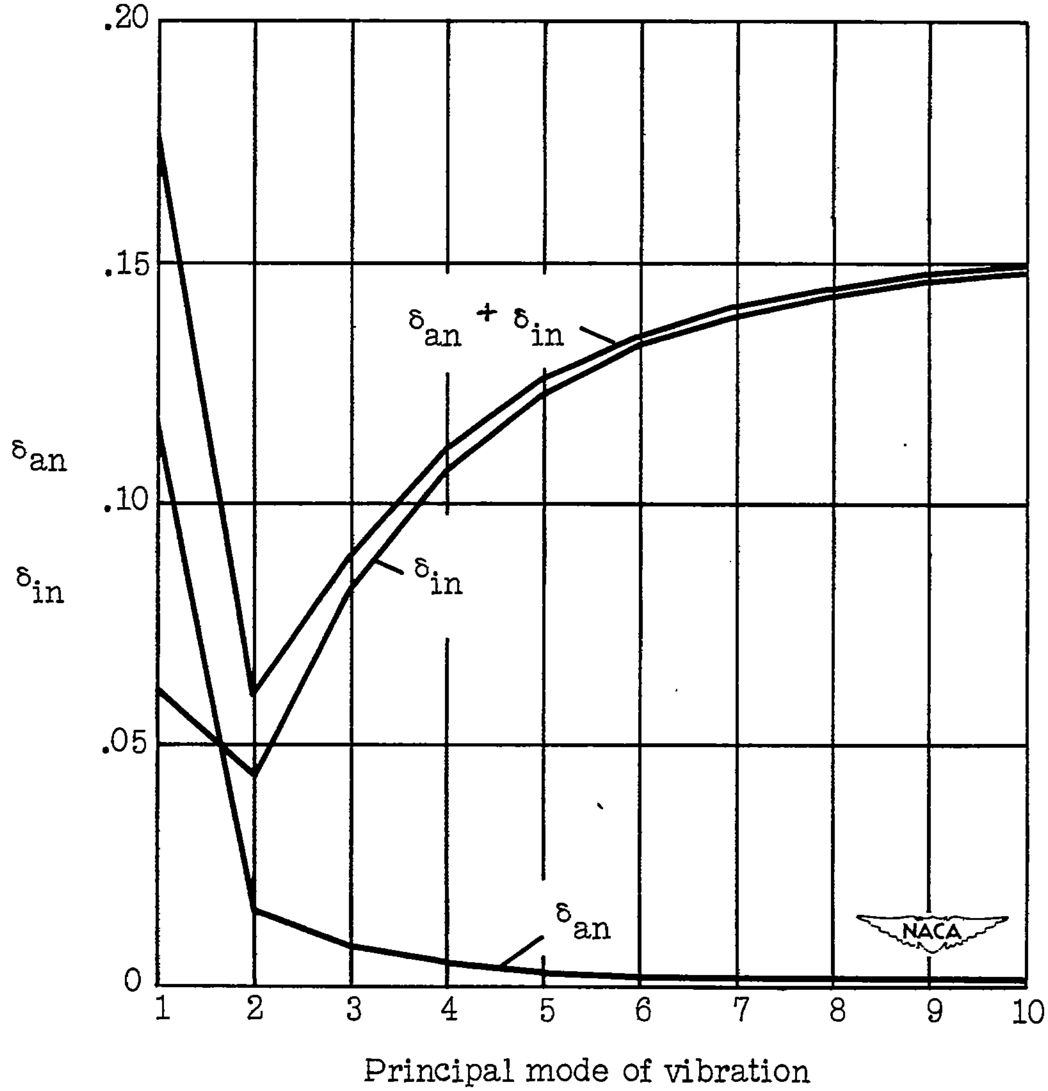


Figure 11.- Aerodynamic and internal logarithmic decrements of fixed-ended blade in lagging. Constant cross section; K_2 = 0.00400; ϵ = 0.05.

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